NOTES

Topological groups, coarse embeddings and quasi-isometry invariants

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Introduction

The aim of these notes is to explore basic concepts and examples in geometric group theory. They are based on several references, the main ones being the books *Metric geometry of locally compact groups* [5], from Yves Cornulier and Pierre de la Harpe, and *Topics in Groups and Geometry* [3] from Tullio Ceccherini-Silberstein and Michele D'Adderio.

Chapter 1 recall several generalities on topological spaces and topological groups. We explain that a $\sigma-$compact locally compact topological group can be endowed with a compatible left-invariant proper metric, that makes it a well-object into the category of pseudo-metric spaces. Therefore, most of our attention is turned to compactly generated locally compact groups, in particular to finitely generated groups.

Chapter 2 then focuses on basic properties of pseudo-metric spaces and maps between them, i.e. coarsely Lipschitz maps and coarse embeddings.

In Chapter 3, we exhibit several examples of such maps, through the fundamental Milnor-Schwarz lemma. We then introduce a first invariant to distinguish groups up to quasi-isometry: the volume growth. We compute the growth for several commonly studied classes of groups, and we wish to emphasize on the idea that the volume growth of a group has a deep relation with its algebraic structure.

Lastly, we investigate in Chapter 4 the notion of coarse simple connectedness, another invariant of metric coarse equivalence. This allows us to explore the class of compactly presented groups.

Contents

1. Generalities on topological groups

1.1 Preliminaries from general topology

The goal of this subsection is to recall some terminologies and results from the more general context of topological spaces.

Definition 1.1. A topological space X is

(i) locally compact if any point $x \in X$ has a compact neighborhood.

(ii) $\sigma-$compact if it is a countable union of compact subspaces.

(iii) first-countable if any point $x \in X$ has a countable basis of neighborhoods.

(iv) second-countable if its topology has a countable basis.

(v) separable if it contains a countable dense subset.

If X is a topological space and $Y \subset X$, Y is called *relatively compact* if its closure $\overline{Y}$ is compact.

A wide class of topological spaces is that of metric spaces.

Definition 1.2. A pseudo-metric on a set X is a map $d_X : X \times X \longrightarrow [0, \infty)$ so that

(i) $d_X(x, x) = 0$ for any $x \in X$.

(ii) $d_X(x, y) = d_X(y, x)$ for any $x, y \in X$.

(iii) $d_X(x, y) \leq d_X(x, z) + d_X(z, y)$ for any $x, y, z \in X$.

Above, the second property is called the *symmetry* of d_X, while the third one is the *triangle inequality*.

A *pseudo-metric space* is a pair (X, d_X), where X is a set and d_X is a pseudo-metric on X. When no confusion is possible, we only write X for the pair (X, d_X).

For a point $x \in X$ and $A \subset X$, we define the *distance from x to A* as

$$d_X(x, A) := \inf_{a \in A} d_X(x, a)$$

and the *diameter* of A as

$$\mathrm{diam}(A) := \sup_{a, a' \in A} d_X(a, a').$$

We say that A is *bounded* if its diameter is finite, and we say that A is *co-bounded* in X if

$$\sup_{x \in X} d_X(x, A) < \infty.$$

Equivalently, A is co-bounded in X if there exists $C > 0$ so that for any $x \in X$, there is $a \in A$ with $d_X(x, a) \leq C$.

For $r > 0$ and $x \in X$, we denote $B_{d_X}(x, r)$ the *open ball centered at* x *of radius* $r > 0$, given by

$$B_{d_X}(x, r) := \{y \in X : d_X(y, x) < r\}.$$

Similarly, $B'_{d_X}(x, r) := \{y \in X : d_X(y, x) \leq r\}$ stands for the *closed ball centered at* x *of radius* $r > 0$.

A *metric* on a set X is a pseudo-metric d_X on X so that $d_X(x, y) = 0$ implies $x = y$. A *metric space* is a pair (X, d_X) where X is a set and d_X is a metric on X. Such a space is canonically a topological space, for the topology τ_{d_X} generated by the collection

$$\{B_{d_X}(x, r) : x \in X, r > 0\}$$

of all open balls of X.

Conversely, given a topological space (X, τ), a metric d_X is said to be *compatible* if $\tau = \tau_{d_X}$. A topological space X is *metrisable* if it has a compatible metric, and *completely metrisable* if it carries a compatible metric for which it is a complete metric space, *i.e.* it carries a compatible metric with respect to which any Cauchy sequence in X converges.

Lastly, a topological space is *Polish* if it is completely metrisable and separable.

Definition 1.3. Let X be a topological space. A pseudo-metric d_X on X is

(i) proper if balls with respect to d_X are relatively compact.

(ii) locally bounded if any point of X has a neighborhood of finite diameter.

(iii) continuous if the map $d_X \colon X \times X \longrightarrow [0, \infty)$ is continuous.

A metric space (X, d_X) is *proper* if its subsets of finite diameter are relatively compact.

Note that if d_X is locally bounded, then any compact subset of X has finite diameter. If X is locally compact, the converse holds, *i.e.* if any compact subset has finite diameter with respect to a pseudo-metric d_X, then d_X is locally bounded.

Proposition 1.4. *Let X be a locally compact space.*

Any proper continuous metric on X is compatible.

Proof. Let thus d_X be a proper continuous metric on X. The map $\mathrm{Id}_X \colon X \longrightarrow (X, \tau_{d_X})$ is bijective and continuous, since d_X is continuous. To conclude, it is enough to show that the image under Id_X of any closed set of X is closed in (X, τ_{d_X}).

Let F be such a closed set, $x \in X$, and let $(x_n)_{n\in\mathbb{N}} \subset F$ be such that $x_n \to x$ in (X, τ_{d_X}) as $n \to \infty$. We must prove that $x \in F$. Since d_X is proper, there exists a compact subset of X containing x_n for all $n \in \mathbb{N}$. Up to extracting a subsequence, we may assume that $(x_n)_{n\in\mathbb{N}}$ converges in X to some point $y \in X$. As F is closed, $y \in F$. Now $(x_n)_{n\in\mathbb{N}}$ also converges to y in (X, τ_{d_X}), whence $y = x$. Hence $x \in F$, which is closed in (X, τ_{d_X}). □

The next result, that we will take for granted, characterise second-countable locally compact spaces.

Theorem 1.5. *Let X be a locally compact space. The following are equivalent.*

(i) *X is second-countable.*

(ii) *X is metrisable and $\sigma-$compact.*

(iii) *X is metrisable and separable.*

(iv) *X is Polish.*

(v) *X has a proper compatible metric.*

Proof. See *e.g.* [1]. □

Here is the last concept we need to recall.

Definition 1.6. A topological space is a Baire space if every countable intersection of dense subsets of X is dense in X.

We conclude with the celebrated Baire's theorem.

Theorem 1.7. *Any complete metric space is a Baire space, and any locally compact topological space is a Baire space.*

1.2 Basic examples and properties

The next definition is one of the most important for the rest of the text.

Definition 1.8. A topological group is a group G endowed with a topology so that the maps $G \times G \longrightarrow G$, $(g,h) \longmapsto gh$ and $G \longrightarrow G$, $g \longmapsto g^{-1}$ are continuous.

Example 1.9. (i) Any group with the discrete topology is a topological group.
(ii) For any $n \geq 1$, $(\mathbb{R}^n,+)$ with its usual Euclidean topology is a topological group. Likewise, $(\mathbb{C}^n,+)$ with its usual topology is a topological group.
(iii) The multiplicative groups $(\mathbb{R}_{>0},\cdot)$ and $(\mathbb{C}^*,\cdot)$ are topological groups, when endowed with their standard topologies.
(iv) If G is a topological group and $H \leqslant G$, then H is itself a topological group when endowed with the subspace topology. For instance, $(S^1,\cdot)$ is a topological group when endowed with the topology induced by the inclusion $S^1 \subset \mathbb{C}^*$.
(v) If G and H are topological groups, then so is $G \times H$ when equipped with the product topology.

Beyond these examples, the class of linear groups is an important source of examples of topological groups. In what follows, for $n \geq 1$, we endow $\mathcal{M}_n(\mathbb{R})$ with the topology coming from the identification

$$\mathcal{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$$

where $\mathbb{R}^{n^2}$ is equipped with its standard topology, induced by the norm $\|\cdot\|_{\mathbb{R}^{n^2}}$.

Equivalently, this is the topology induced by the operator norm $\|\cdot\|_{\mathrm{op}}\colon \mathcal{M}_n(\mathbb{R}) \longrightarrow [0,+\infty)$, defined as

$$\|g\| := \sup_{x\neq 0} \frac{\|gx\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} = \sup_{\|x\|_{\mathbb{R}^n}\leq 1} \|gx\|_{\mathbb{R}^n} = \sup_{\|x\|_{\mathbb{R}^n}=1} \|gx\|_{\mathbb{R}^n}$$

where $\|\cdot\|_{\mathbb{R}^n}$ denotes the usual Euclidean norm on $\mathbb{R}^n$, and gx denotes the multiplication of the $n\times n$ matrix g with the $n\times 1$ matrix x.

The fact that these two topologies coincide follows from the equivalence of the norms $\|\cdot\|_{\mathbb{R}^{n^2}}$ and $\|\cdot\|_{\mathrm{op}}$, and the equivalence of these two norms is a general fact valid for any finite dimensional vector space [2].

Proposition 1.10. *The general linear group*

$$\mathrm{GL}_n(\mathbb{R}) := \{g \in \mathcal{M}_n(\mathbb{R}) : \det(g) \neq 0\}$$

is a topological group.

In this statement and in the rest of this text, $\mathrm{GL}_n(\mathbb{R})$ is endowed with the topology induced by the inclusion $\mathrm{GL}_n(\mathbb{R}) \subset \mathcal{M}_n(\mathbb{R})$.

To prove the previous proposition, we first establish the following fact.

Lemma 1.11. *The subspace* $\mathrm{GL}_n(\mathbb{R})$ *is open in* $\mathcal{M}_n(\mathbb{R})$.

Proof. The lemma will be a direct consequence of the next claim:
Let $M \in \mathcal{M}_n(\mathbb{R})$. If $\|M\| < 1$, then $I_n - M$ is invertible and $\|(I_n - M)^{-1}\| \leq \frac{1}{1-\|M\|}$.

Indeed, for any $n \geq 0$, define $S_k := \sum_{i=0}^{n} M^i$. Then, for $k \geq \ell \geq 0$, one gets

$$\|S_k - S_\ell\| = \left\| \sum_{i=\ell+1}^{k} M^i \right\| \leq \sum_{i=\ell+1}^{k} \|M\|^i$$

and this last quantity tends to 0 as $k,\ell \to \infty$ since $\|M\| < 1$. Hence $(S_k)_{k\in\mathbb{N}}$ is Cauchy in $\mathcal{M}_n(\mathbb{R})$, which is complete, and thus converges to $T \in \mathcal{M}_n(\mathbb{R})$. A direct computation shows that

$$T(I_n - M) = (I_n - M)T = I_n$$

whence in fact $T = (I_n - M)^{-1}$, and also

$$\|T\| = \|(I_n - M)^{-1}\| = \left\| \lim_{k\to\infty} \sum_{i=0}^{k} M^i \right\| \leq \lim_{k\to\infty} \sum_{i=0}^{k} \|M\|^i = \frac{1}{1-\|M\|}$$

and the claim is proved. Now let $A \in \mathrm{GL}_n(\mathbb{R})$. If $B \in \mathcal{M}_n(\mathbb{R})$ is so that $\|B - A\| < \frac{1}{\|A^{-1}\|}$, then

$$\|I_n - A^{-1}B\| = \|A^{-1}(A - B)\| \leq \|A^{-1}\|\|A - B\| < 1$$

whence $I_n - (I_n - A^{-1}B) = A^{-1}B$ is invertible, by the claim. It follows that $B = A(A^{-1}B)$ is invertible as the product of two invertible matrices, *i.e.* $B \in \mathrm{GL}_n(\mathbb{R})$. Hence $\mathrm{GL}_n(\mathbb{R})$ is a neighborhood of any of its elements, which means it is open in $\mathcal{M}_n(\mathbb{R})$. □

Proof of Proposition 1.10. If $\varepsilon > 0$ and $A, B \in \mathrm{GL}_n(\mathbb{R})$ are so that $\|B - A\| \leq \varepsilon$, then $\|A^{-1}(A - B)\| \leq \|A^{-1}\|\varepsilon$, and as $B^{-1} = (I_n - A^{-1}(A - B))^{-1}A^{-1}$, we get

$$\begin{aligned}\|B^{-1}\| &\leq \|(I_n - A^{-1}(A-B))^{-1}\|\|A^{-1}\| \\ &\leq \frac{\|A^{-1}\|}{1 - \|A^{-1}(A-B)\|} \\ &\leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\varepsilon}.\end{aligned}$$

It follows that

$$\begin{aligned}\|B^{-1} - A^{-1}\| &= \|B^{-1}(A-B)A^{-1}\| \\ &\leq \|B^{-1}\|\|(A-B)A^{-1}\| \\ &\leq \frac{\|A^{-1}\|^2\varepsilon}{1 - \|A^{-1}\|\varepsilon}\end{aligned}$$

whence $B \longmapsto B^{-1}$ is continuous at $A \in \mathrm{GL}_n(\mathbb{R})$. Hence $\mathrm{GL}_n(\mathbb{R})$ is a topological group. □

Likewise, $\mathrm{GL}_n(\mathbb{C})$ is a topological group.

The next proposition give examples of compact subsets of $\mathrm{GL}_n(\mathbb{R})$.

Proposition 1.12. *For any $C > 0$, the subset*

$$Q_C := \{g \in \mathrm{GL}_n(\mathbb{R}) : \|g\| \leq C, \|g^{-1}\| \leq C\}$$

is compact.

Proof. Since the topology on $\mathrm{GL}_n(\mathbb{R})$ is induced by a norm, compactness is equivalent to sequential compactness, and it is enough to prove that any sequence in Q_C has a convergent subsequence.

Let thus $(g_j)_{j\in\mathbb{N}} \subset Q_C$ be such a sequence. Then $(g_j)_{j\in\mathbb{N}}$ lies in the closed ball of radius $C > 0$ and centered at 0 (the zero matrix) in $\mathcal{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, and such closed balls are compact, so $(g_j)_{j\in\mathbb{N}}$ has a convergent subsequence $(g_{j_k})_{k\in\mathbb{N}}$. Calling $g \in \mathcal{M}_n(\mathbb{R})$ the limit of $(g_{j_k})_{k\in\mathbb{N}}$, it follows that $\|g\| \leq C$. Now $\|g_{j_k}^{-1}\| \leq C$ for any $k \in \mathbb{N}$, so we can argue as above to find a subsequence $(g_{j_{k_l}}^{-1})_{l\in\mathbb{N}}$ of $(g_{j_k}^{-1})_{k\in\mathbb{N}}$ converging to some $h \in \mathcal{M}_n(\mathbb{R})$. As $\|g_{j_{k_l}}^{-1}\| \leq C$ for every $l \in \mathbb{N}$, we also deduce $\|h\| \leq C$. Lastly, from the fact that

$$g_{j_{k_l}} g_{j_{k_l}}^{-1} = I_n$$

for any $l \in \mathbb{N}$, we get $gh = I_n$, so $g \in \mathrm{GL}_n(\mathbb{R})$ and $h = g^{-1}$. Thus $(g_j)_{j\in\mathbb{N}}$ has a subsequence converging to $g \in Q_C$, which concludes the proof. □

In particular, it follows from this result that $\mathrm{GL}_n(\mathbb{R}) = \bigcup_{n\in\mathbb{N}} Q_n$ is σ−compact.

Definition 1.13. A morphism of topological groups is a continuous group homomorphism $f : G \longrightarrow H$, and an isomorphism of topological groups is a bijective morphism $f : G \longrightarrow H$ of topological groups whose inverse $f^{-1} : H \longrightarrow G$ is also a morphism of topological groups, *i.e.* is also continuous.

When there is an isomorphism between two topological groups G and H, we say they are *isomorphic*, and we denote $G \cong H$.

Subgroups of $\mathrm{GL}_n(\mathbb{R})$ also provide interesting examples of topological groups.

Example 1.14. (i) The special linear group

$$\mathrm{SL}_n(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) : \det(g) = 1\}$$

is a topological group for the subspace topology induced by the inclusion $\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$. It is a closed and normal in $\mathrm{GL}_n(\mathbb{R})$, since $\mathrm{SL}_n(\mathbb{R}) = \det^{-1}(\{1\})$ and since $\det\colon \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R}^*$ is continuous.

(ii) The orthogonal group

$$\mathrm{O}_n(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) : g^T g = g g^T = I_n\}$$

is a topological group. It is compact as a consequence of Proposition 1.12, since for $g \in \mathrm{O}_n(\mathbb{R})$, $\|g\| = \|g^{-1}\| = 1$.

(iii) The special orthogonal group

$$\mathrm{SO}_n(\mathbb{R}) := \mathrm{O}_n(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})$$

is a topological group, compact as it is closed in $\mathrm{O}_n(\mathbb{R})$ which is compact. For $n = 2$, the group $\mathrm{SO}_2(\mathbb{R})$ is isomorphic to the unit circle, through the isomorphism

$$\begin{aligned} \mathrm{SO}_2(\mathbb{R}) &\longrightarrow S^1 \\ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} &\longmapsto \cos(\theta) + i\sin(\theta). \end{aligned}$$

(iv) Let us mention the upper triangular group

$$\mathrm{T}_n(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) : g_{ij} = 0 \text{ if } i > j\}$$

and the strict upper triangular group

$$\mathrm{T}_0(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) : g_{ij} = 0 \text{ if } i > j,\ g_{ii} = 1\}.$$

It is easily checked that $\mathrm{T}_0(\mathbb{R})$ is normal in $\mathrm{T}_n(\mathbb{R})$.

Remark 1.15. Let G be a topological group, and $g \in G$. The maps $G \longrightarrow G$, $x \longmapsto g$ and $G \longrightarrow G$, $x \longmapsto x$ are continuous, so the map $G \longrightarrow G \times G$, $x \longmapsto (g,x)$ is continuous. Composing with the multiplication, it follows that the map $\ell_g \colon G \longrightarrow G$, $\ell_g(x) = gx$, called the *left translation by* g, is continuous. It is also a bijection, whose inverse $\ell_{g^{-1}}$ is also continuous. Thus ℓ_g is a homeomorphism. Likewise, the *right translation by* g $r_g \colon G \longrightarrow G$, $r_g(x) = xg$, is a homeomorphism. Thus G is a *homogeneous space, i.e.* given any $a, b \in G$, there is a homeomorphism $G \longrightarrow G$ sending a to b. This means that topologically, G "looks the same" at all points.

Let G be a topological group, $A, B \subset G$, and $g \in G$. Then we denote $Ag := \{ag : a \in A\}$, $gA := \{ga : a \in A\}$, $A^{-1} := \{a^{-1} : a \in A\}$ and

$$AB := \{ab : a \in A, b \in B\} = \bigcup_{a \in A} aB = \bigcup_{b \in B} Ab.$$

A subset $A \subset G$ is called *symmetric* if $A^{-1} = A$.

The following observations are straightforward from the definitions.

Lemma 1.16. *Let G be a topological group, $A, B \subset G$, $g \in G$.*

(i) *If A is open (resp. closed), then gA, Ag are open (resp. closed).*

(ii) *If A is open, then AB, BA are open.*

(iii) *If A is closed and B is finite, then AB, BA are closed.*

Proof. (i) Suppose A is open. Then $gA = \ell_g(A)$, $Ag = r_g(A)$ are open since ℓ_g, r_g are homeomorphisms (so in particular, these two maps are open).

(ii) If A is open, then Ab is open for any $b \in B$ by (i), so AB is open as a union of open sets. Likewise, bA is open for any $b \in B$, whence BA is open as well.

(iii) If now A is closed, then Ab, bA are closed for any $b \in B$, and thus AB, BA are closed as finite unions of closed sets. □

If G is a topological group, and $H \leqslant G$, G/H is a topological space, with $U \subset G/H$ open if and only if $q^{-1}(U) \subset G$ open, where $q \colon G \longrightarrow G/H$ is the natural surjection. This map is always open, because if $S \subset G$ is open, then $q^{-1}(q(S)) = SH$ is the union of all left H−cosets meeting S. Lemma 1.16 ensures that SH is open since S open, so $q(S)$ is open in G/H, as claimed. If in addition $H \lhd G$, then G/H is a topological group, with the usual universal property: if $f : G \longrightarrow L$ is a morphism of topological groups with $H \leqslant \mathrm{Ker}(f)$, there exists a unique morphism $f^* \colon G/H \longrightarrow L$ of topological groups so that $f^* \circ q = f$. In particular, a morphism of topological groups $f \colon G \longrightarrow L$ always induces an isomorphism of topological groups

$$G/\mathrm{Ker}(f) \cong \mathrm{Im}(f).$$

Example 1.17. (i) The map $f : \mathbb{R} \longrightarrow S^1$, $t \longmapsto e^{2\pi i t}$ is a surjective morphism of topological groups, with $\mathrm{Ker}(f) = \mathbb{Z}$. Hence $\mathbb{R}/\mathbb{Z} \cong S^1$ as topological groups.

(ii) The morphism $f\colon \mathbb{C}^* \longrightarrow \mathbb{R}_{>0}$, $z \longmapsto |z|$ induces an isomorphism $\mathbb{C}^*/S^1 \cong \mathbb{R}_{>0}$.

(iii) The map $\det\colon \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R}^+$ is a surjective morphism of topological groups with kernel $\mathrm{SL}_n(\mathbb{R})$. Thus $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$.

Proposition 1.18. *Let G be a topological, and let $\mathcal{F}$ be a basis of open neighborhoods of $e \in G$. The following properties hold.*

(i) *If $U, V \in \mathcal{F}$, then there exists $W \in \mathcal{F}$ with $W \subset U \cap V$.*

(ii) *If $a \in U \in \mathcal{F}$, then there exists $V \in \mathcal{F}$ with $Va \subset U$.*

(iii) *If $U \in \mathcal{F}$, then there exists $V \in \mathcal{F}$ with $V^{-1}V \subset U$.*

(iv) *If $U \in \mathcal{F}$ and $x \in G$, then there exists $V \in \mathcal{F}$ with $x^{-1}Vx \subset U$.*

Proof. (i) Let $U, V \in \mathcal{F}$. Then $U \cap V$ is an open neighborhood of e, so there is $W \in \mathcal{F}$ so that $W \subset U \cap V$.

(ii) If $U \in \mathcal{F}$ and $a \in U$, then Ua^{-1} is an open neighborhood of $e \in G$, so there is $V \in \mathcal{F}$ with $V \subset Ua^{-1}$, equivalently $Va \subset U$.

(iii) Let $U \in \mathcal{F}$, and consider the map $f\colon G\times G \longrightarrow G$, $(a,b) \longmapsto a^{-1}b$. As G is a topological group, f is continuous, and as $(e,e) \in f^{-1}(U)$, there exist two open subsets $A, B \subset G$ both containing e and so that $A \times B \subset f^{-1}(U)$. Hence $A \cap B$ is an open neighborhood of e, so there is $V \in \mathcal{F}$ with $V \subset A \cap B \subset f^{-1}(U)$, whence $V^{-1}V \subset U$.

(iv) Let now $U \in \mathcal{F}$ and $x \in G$. Consider the map $f\colon G \longrightarrow G$, $g \longmapsto x^{-1}gx$. As above, f is continuous, so $f^{-1}(U)$ is open, and contains $e \in G$. Hence there is $V \in \mathcal{F}$ with $V \subset f^{-1}(U)$, *i.e.* $x^{-1}Vx \subset U$. □

Definition 1.19. Let X be a topological space.

(i) The space X is a T_1–space if for any $x, y \in X$, $x \neq y$, there is a neighborhood of x not containing y.

(ii) The space X is a T_2–space (or Hausdorff) if for any $x, y \in X$, $x \neq y$, there exist neighborhoods of x and y that are disjoint.

Remark 1.20. (i) Clearly, if X is Hausdorff, then X is T_1.

(ii) A space X is T_1 if and only if $\{x\}$ is closed for all $x \in X$.

The next result is standard in general topology. We include the proof for completeness.

Proposition 1.21. *Let X be a topological space. The following are equivalent.*

(i) *The space X is Hausdorff.*

(ii) *The diagonal $\Delta_X := \{(x,x) \in X \times X : x \in X\}$ is closed in $X \times X$.*

(iii) *For any topological space Y and any continuous functions $f, g\colon Y \longrightarrow X$, the set*

$$Z := \{y \in Y : f(y) = g(y)\}$$

is closed in Y.

Proof. (i) $\Longrightarrow$ (iii) : Let Y be a topological space and $f, g\colon Y \longrightarrow X$ be continuous maps. Let $y \in Y \setminus Z$. Thus $f(y) \neq g(y)$, and since X is Hausdorff, we find $U, V \subset X$ open so that $f(y) \in U$, $g(y) \in V$ and $U \cap V = \emptyset$. It follows that $f^{-1}(U)$, $g^{-1}(V)$ are open in Y since f and g are continuous, and they both contain y, so $f^{-1}(U) \cap g^{-1}(V)$ is also open in X and also contains y. It remains to notice that $f^{-1}(U) \cap g^{-1}(V) \subset Y \setminus Z$ to conclude that $Y \setminus Z$ is a neighborhood of any of its elements, *i.e.* is open in Y, and thus Z is closed.

(iii) $\Longrightarrow$ (ii) : It is enough to apply (iii) with $Y = X \times X$ and f, g the projections on the first and second factor respectively (which are both continuous) to conclude that the diagonal is closed in $X \times X$.

(ii) $\Longrightarrow$ (i) : Let $x, y \in X$, $x \neq y$. Then $(x, y) \in (X \times X) \setminus \Delta_X$, which is open by assumption, so we can find two open subsets $U, V \subset X$ with $(x, y) \in U \times V \subset (X \times X) \setminus \Delta_X$. Thus $x \in U$, $y \in V$, and the intersection $U \cap V$ is empty since a point $z \in U \cap V$ would provide a point (z, z) of the product $U \times V$ and of the diagonal Δ_X, contradicting the fact that these two sets are disjoint. Hence X is Hausdorff. □

This proposition allows us to deduce the next one for topological groups.

Proposition 1.22. *Let G be a topological group, and let $\mathcal{F}$ be a basis of neighborhoods of $e \in G$. The following are equivalent.*

(i) *The group G is Hausdorff.*

(ii) *The diagonal $\Delta_G = \{(g, h) \in G \times G : g = h\}$ is closed in $G \times G$.*

(iii) *For any topological group H and any morphisms $f, g\colon H \longrightarrow G$, the subset*

$$Z_{f,g} = \{h \in H : f(h) = g(h)\}$$

is a closed subgroup of H.

(iv) *For any topological group H and any morphism $f\colon H \longrightarrow G$, Ker$(f)$ is a closed subgroup of H.*

(v) *The subgroup $\{e\}$ is closed in G.*

(vi) *The group G is T_1.*

(vii) $\displaystyle\bigcap_{F \in \mathcal{F}} F = \{e\}$.

(viii) *The intersection of all neighborhoods of e is $\{e\}$.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) follows from Proposition 1.21, and additionally in (iii) the subset $Z_{f,g}$ is easily checked to be a subgroup.

(iii) $\Longrightarrow$ (iv) : Fix a morphism $f\colon H \longrightarrow G$ and define $g\colon H \longrightarrow G$, $h \longmapsto e$. The map g is a morphism of topological groups, and in this case the set $Z_{f,g}$ is nothing but the kernel of f, so from (iii) we deduce that Ker(f) is closed in H.

(iv) $\Longrightarrow$ (v) : Apply (iv) with $H = G$ and $f = \mathrm{Id}_G$ to deduce that Ker$(f) = \{e\}$ is closed in G.

(v) $\Longrightarrow$ (vi) : To prove G is T_1, it is enough to prove that $\{g\}$ is closed for any $g \in G$, by Remark 1.20. For any $g \in G$, $\{g\} = g\{e\} = \ell_g(\{e\})$ and the closedness of $\{e\}$ implies that $\{g\}$ is also closed, since ℓ_g is a homeomorphism.

(vi) $\Longrightarrow$ (vii) : The inclusion of $\{e\}$ in $\bigcap_{F\in\mathcal{F}} F$ is obvious. Conversely, if $g \in G$, $g \neq e$, then $G \setminus \{g\}$ is open by (vi) and contains e. Hence there exists $F \in \mathcal{F}$ so that $e \in F \subset G \setminus \{g\}$, whence $g \notin \bigcap_{F\in\mathcal{F}} F$. This proves the desired equality.

(vii) $\Longrightarrow$ (viii) is obvious.

(viii) $\Longrightarrow$ (i) : Let $g, h \in G$, $g \neq h$. Then $gh^{-1} \neq e$, so by (viii) there exists a neighborhood U of e so that $gh^{-1} \in G \setminus U$, and without restriction we may assume that $U \in \mathcal{F}$. Thus, Proposition 1.18(iii) ensures that there exists $V \in \mathcal{F}$ so that $V^{-1}V \subset U$. In particular, Vgh^{-1} is a neighborhood of gh^{-1}, and as $V^{-1}V \subset U$, it follows that

$$Vgh^{-1} \cap V = \emptyset.$$

Equivalently, $Vg \cap Vh = \emptyset$, whence Vg and Vh are disjoint neighborhoods of g and h respectively. It follows that G is Hausdorff. □

These equivalent characterisations of the Hausdorff property allow us to prove much more easily stability properties among the class of topological groups.

Proposition 1.23. *Let G be a topological group, and $H \leqslant G$.*

(i) *If G is Hausdorff, then H is Hausdorff.*

(ii) *The quotient G/H is Hausdorff if and only if H is closed in G.*

(iii) *If H, G/H are Hausdorff, then G is Hausdorff.*

Proof. (i) is already true in the more general context of topological spaces.

(ii) Suppose first that G/H is Hausdorff, and denote $q \colon G \longrightarrow G/H$ the canonical projection. In particular, G/H is T_1, so the singleton $\{q(e)\}$ is closed in G/H. Thus $H = q^{-1}(\{q(e)\})$ is closed in G as q is continuous.

Conversely, suppose H is closed in G, and let aH, bH be two disctinct elements of G/H. As q is open and continuous, it is enough to find W a neighborhood of e in G so that $WaH \cap WbH = \emptyset$. Since $a^{-1}bH$ is closed (as the left translation by $a^{-1}b$ is a homeomorphism) and does not contain e, so we find a neighborhood U of e in G so that $U \cap a^{-1}bH = \emptyset$. Now, Proposition 1.18(iii), (iv) shows we can find neighborhoods V, W of e so that $V^{-1}V \subset U$, $a^{-1}Wa \subset V$. Now $WaH \cap WbH = \emptyset$.

(iii) As H is Hausdorff, $\{e\}$ is closed in H by Proposition 1.22. As G/H is Hausdorff, H is closed in G by (ii). Thus $\{e\}$ is closed in G, whence G is Hausdorff, again by Proposition 1.22. □

Conversely, open subgroups of topological groups provide quotient spaces far from being Hausdorff.

Proposition 1.24. *Let G be a topological group, and $H \leqslant G$.*

(i) *If H is open, then H is closed. Conversely, if H is closed and has finite index, then H is open.*

(ii) *If H contains a neighborhood of e, then H is open.*

(iii) *The quotient G/H is discrete if and only if H is open in G.*

Proof. (i) If H is open, then so are all left H−cosets in G, whence

$$G \setminus H = \bigcup_{g \notin H} gH$$

is open in G. Thus H is closed in G.

Conversely, if H is closed and has finite index, then $G \setminus H$ is the union of finitely many left H−cosets, thus $G \setminus H$ is also closed in G. Therefore, H is open in G.

(ii) If H contains a neighborhood of e, then H contains U an open neighborhood of e, and thus $H = HU$ is open by Lemma 1.16(ii).

(iii) If G/H is discrete, then $\{q(e)\}$ is open, so $H = q^{-1}(\{q(e)\})$ is open in G as the natural surjection $q\colon G \longrightarrow G/H$ is continuous.

Conversely, any singleton in G/H is the image under q of a left H−coset. Such a left coset is open in G, thus any singleton in G/H is open as q is an open map. □

For the next statement, recall that a topological space X is connected if it is non-empty and cannot be decomposed as the disjoint union of two non-empty open subsets. Equivalently, X is connected if the only subsets of X that are both open and closed are $\emptyset$ and X itself.

Example 1.25. (i) A discrete group is connected if and only if it is reduced to one element.

(ii) For any $n \geq 1$, $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$ are connected.

(iii) The multiplicative groups $(\mathbb{R}_{>0}, \cdot)$, $(\mathbb{C}^*, \cdot)$, $(S^1, \cdot)$ are connected.

(iv) For any $n \geq 1$, the general linear group $\mathrm{GL}_n(\mathbb{R})$ is not connected, since the determinant is continuous and that $\mathbb{R}^*$ is not connected. The same reason and the fact that $\{-1, 1\}$ is not connected shows that $\mathrm{O}_n(\mathbb{R})$ is not connected either.

Corollary 1.26. (i) *A connected topological group has no proper open subgroups.*

(ii) *A connected topological group is generated by any neighborhood of the identity.*

Proof. (i) If G is connected, and H is a proper open subgroup, then H is also closed (by Proposition 1.24(i)), a contradiction with the connectedness of G.

(ii) If U is such a neighborhood, let $H := \langle U \rangle$. From point (ii) of the previous proposition, H is open, hence closed, and thus $H = G$ by connectedness of G. □

We finish this subsection with the following stability property for connected topological groups.

Proposition 1.27. *Let G be a topological group and let $H \leqslant G$.*

(i) *If G is connected, then G/H is connected.*

(ii) *If $H, G/H$ are connected, then G is connected.*

Proof. (i) is already true for topological spaces.

(ii) For a contradiction, assume that $G = A \sqcup B$ where A, B are non-empty open subsets of G. Since H is connected, so are all its left cosets. Hence each coset must be contained either in A or in B, so A and B are union of left $H-$cosets. If $q\colon G \longrightarrow G/H$ is the quotient map, then

$$G/H = q(A) \sqcup q(B)$$

and A, B are both open and closed in G, so $q(A)$ and $q(B)$ are both open and closed in G/H. It follows that G/H is disconnected, a contradiction. Hence G is connected. □

For $n \geq 1$, let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_{\mathbb{R}^n} = 1\}$ be the unit sphere in $\mathbb{R}^n$. The general linear group $\mathrm{GL}_n(\mathbb{R})$ acts naturally on $\mathbb{R}^n$, and thus so does $\mathrm{SO}_n(\mathbb{R})$. This action preserves S^{n-1} (as any orthogonal matrix acts by isometries on $\mathbb{R}^n$), so restricts to an action $\mathrm{SO}_n(\mathbb{R}) \curvearrowright S^{n-1}$.

Proposition 1.28. *For any $n \geq 2$, $\mathrm{SO}_n(\mathbb{R}) \curvearrowright S^{n-1}$ is transitive.*

Proof. We prove the statement by induction on $n \geq 2$. If $n = 2$, $\mathrm{SO}_2(\mathbb{R})$ is the group of rotations of the plane, that acts transitively on the unit circle S^1.

Now suppose the statement holds up to $n-1$. To prove that $\mathrm{SO}_n(\mathbb{R}) \curvearrowright S^{n-1}$ is transitive, it is enough to prove that for any $x \in S^{n-1}$, there exists $k \in \mathrm{SO}_n(\mathbb{R})$ so that $x = ke_n$, where $e_n = (0, \ldots, 0, 1)$. Fix such a point $x \in S^{n-1}$. Then we can write

$$x = \cos(\theta)e_n + \sin(\theta)x'$$

with x' is in the subspace spanned by $e_1, \ldots, e_{n-1}$, the $n-1$ first vectors of the canonical basis, and has $\|x'\|_{\mathbb{R}^{n-1}} = 1$. In other words, $x \in S^{n-2}$. By the induction hypothesis, there exists $k' \in \mathrm{SO}_{n-1}(\mathbb{R})$ with $x' = k'e_{n-1}$. Set then

$$u := \begin{pmatrix} k' & 0 \\ 0 & 1 \end{pmatrix},\ h_\theta := \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Then it follows that

$$uh_\theta e_n = \sin(\theta)k'e_{n-1} + \cos(\theta)e_n = \cos(\theta)e_n + \sin(\theta)x' = x$$

and we choose $k := uh_\theta$. This concludes the inductive step and the proof. □

If we denote $K := \mathrm{Stab}(e_n) = \{k \in \mathrm{SO}_n(\mathbb{R}) : ke_n = e_n\}$ the stabilizer of $e_n \in S^{n-1}$, we have that

$$K = \left\{ \begin{pmatrix} k' & 0 \\ 0 & 1 \end{pmatrix} : k' \in \mathrm{SO}_{n-1}(\mathbb{R}) \right\} \cong \mathrm{SO}_{n-1}(\mathbb{R})$$

and we deduce the following corollary.

Corollary 1.29. (i) *Any* $k \in \mathrm{SO}_n(\mathbb{R})$ *can be written* $k = k_1 h_\theta k_2$ *with* $k_1, k_2 \in K \cong \mathrm{SO}_{n-1}(\mathbb{R})$ *and* $\theta \in \mathbb{R}$.

(ii) *For any* $n \geq 2$, $\mathrm{SO}_n(\mathbb{R})$ *is connected.*

Proof. (i) Let $k \in \mathrm{SO}_n(\mathbb{R})$, and let $x = ke_n$. According to the proof of the previous proposition, we may write $x = k' h_\theta$, with $k' \in K$. Thus $h_\theta^{-1} k_1^{-1} k e_n = e_n$, whence $k_2 := h_\theta^{-1} k_1^{-1} k \in K$, and $k = k_1 h_\theta k_2$.

(ii) We prove the statement by induction on $n \geq 2$. For $n = 2$, $\mathrm{SO}_2(\mathbb{R})$ is isomorphic to the unit circle, hence is connected.

Now assume that $\mathrm{SO}_{n-1}(\mathbb{R})$ is connected for some $n \geq 2$. From (i), there is a continuous surjection

$$\mathrm{SO}_{n-1}(\mathbb{R}) \times \mathbb{R} \times \mathrm{SO}_{n-1}(\mathbb{R}) \longrightarrow \mathrm{SO}_n(\mathbb{R}),\ (k_1, \theta, k_2) \longmapsto k_1 h_\theta k_2.$$

Since $\mathrm{SO}_{n-1}(\mathbb{R}) \times \mathbb{R} \times \mathrm{SO}_{n-1}(\mathbb{R})$ is connected, it follows that $\mathrm{SO}_n(\mathbb{R})$ is connected, and the proof is complete. □

In fact, the same proof shows that $\mathrm{SO}_n(\mathbb{R})$ is path-connected for any $n \geq 2$, in particular connected.

Recall that any $g \in \mathrm{GL}_n(\mathbb{R})$ has a unique *polar decomposition, i.e.* a decomposition $g = kp$ with $k \in \mathrm{O}_n(\mathbb{R})$, $p \in \mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$ denote the subset of positive definite symmetric $n \times n$ matrices. It is an open convex cone in the vectorspace of real symmetric $n \times n$ matrices. Moreover, the map

$$\mathrm{O}_n(\mathbb{R}) \times \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathrm{GL}_n(\mathbb{R}),\ (k, p) \longmapsto kp$$

is a homeomorphism. More details on this result can be found in [8, theorem 1.4.1] for the finite-dimensional case over the field of real numbers, and in [7, theorem 1.46] for the general case over the field of complex numbers.

Denoting $\mathrm{GL}_n^+(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) : \det(g) > 0\}$, it follows that any $g \in \mathrm{GL}_n^+(\mathbb{R})$ can be uniquely written as $g = kp$ where $k \in \mathrm{SO}_n(\mathbb{R})$ and $p \in \mathcal{P}(\mathbb{R}^n)$, and that the map

$$\mathrm{SO}_n(\mathbb{R}) \times \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathrm{GL}_n^+(\mathbb{R}),\ (k, p) \longmapsto kp$$

is a homeomorphism. As $\mathrm{SO}_n(\mathbb{R})$ and $\mathcal{P}(\mathbb{R}^n)$ are both connected, we deduce the next result.

Corollary 1.30. *For any* $n \geq 1$, *the topological groups* $\mathrm{GL}_n^+(\mathbb{R})$ *and* $\mathrm{SL}_n(\mathbb{R})$ *are connected.*

1.3 Metrisation theorems for topological groups

This section is devoted to establish several metrisation theorems for topological groups. From now on, any topological space is assumed to be Hausdorff, unless mentioned otherwise.

The starting point is the following lemma.

Lemma 1.31. *Let G be a topological group. Assume there exists a sequence $(K_n)_{n\in\mathbb{Z}}$ of subsets of G so that*

(i) $G = \bigcup_{n\in\mathbb{Z}} K_n$.

(ii) *For any $n \in \mathbb{Z}$, K_n is symmetric and $e \in K_n$.*

(iii) *$K_nK_nK_n \subset K_{n+1}$ for any $n \in \mathbb{Z}$.*

(iv) *There exists $m \in \mathbb{Z}$ so that $K_m^\circ \neq \emptyset$.*

Define a map $d\colon G \times G \longrightarrow [0, +\infty)$ by

$$d(g,h) := \inf\Big\{t \geq 0 : \exists n_1,\dots,n_k \in \mathbb{Z},\ \exists w_1 \in K_{n_1},\dots,w_k \in K_{n_k} \text{ so that } g^{-1}h = w_1\dots w_k,\, t = 2^{n_1}+\dots+2^{n_k}\Big\}$$

and write $|g| = d(e,g)$. Then the following claims hold.

(i) *The map d is a left-invariant pseudo-metric on G for which every compact subset of G has finite diameter.*

(ii) *If K_n is a neighborhood of e for any $n \in \mathbb{Z}$, then d is continuous.*

(iii) *For any $n \in \mathbb{Z}$, if $|g| < 2^n$ then $g \in K_n$.*

(iv) *One has $\bigcap_{n\in\mathbb{Z}} K_n = \{g \in G : |g| = 0\}$. In particular, if $\bigcap_{n\in\mathbb{Z}} K_n = \{e\}$, then d is a metric.*

Proof. (i) First let $g \in G$. Then $g^{-1}g = e \in K_n$ for any $n \in \mathbb{Z}$, hence $d(g,g) \leq 2^n$ for any $n \in \mathbb{Z}$. This implies $d(g,g) = 0$. For the symmetry, note that if $g, h \in G$ and if $g^{-1}h = w_1 \dots w_k$ for some $w_1 \in K_{n_1},\dots,w_k \in K_{n_k}$, then

$$h^{-1}g = (g^{-1}h)^{-1} = w_{n_k}^{-1}\dots w_{n_1}^{-1}$$

and $w_{n_i}^{-1} \in K_{n_i}$ for any $i = 1,\dots,k$, as any K_n is symmetric. Thus

$$d(h,g) \leq 2^{n_1}+\dots+2^{n_k}$$

and it follows that $d(h,g) \leq d(g,h)$. The reverse inequality follows swapping the roles of g and h, whence $d(h,g) = d(g,h)$ for all $g, h \in G$, and d is symmetric. Lastly, if $g, h, a \in G$, the triangle inequality for g, h, a follows from the fact that if

$$g^{-1}a = w_1\dots w_k,\ a^{-1}h = v_1\dots v_r$$

for some $n_1,\dots,n_k, m_1,\dots,m_r \in \mathbb{Z}$ and some group elements $w_1 \in K_{n_1},\dots,w_k \in K_{n_k}, v_1 \in K_{m_1},\dots v_r \in K_{m_r}$, then

$$g^{-1}h = (g^{-1}a)(a^{-1}h) = w_1\dots w_kv_1\dots v_r$$

exhibits $g^{-1}h$ as a product of $k+r$ group elements, so we deduce that

$$d(g,h) \leq 2^{n_1} + \cdots + 2^{n_k} + 2^{m_1} + \cdots + 2^{m_r}$$

and thus that $d(g,h) \leq d(g,a) + d(a,h)$. Therefore d is a pseudo-metric on G.

Its left-invariance is also clear: if $a, g, h \in G$ and $g^{-1}h = v_1 \dots v_k$ with $v_1 \in K_{n_1}, \dots, v_k \in K_{n_k}$, then

$$(ag)^{-1}(ah) = g^{-1}h = v_1 \dots v_k$$

as well, so $d(ag, ah) = d(g,h)$.

Now let $L \subset G$ be compact. Since K_m has non-empty interior, K_{m+1} is a neighborhood of e, and $L \subset \bigcup_{\ell \in L} \ell K^{\circ}_{m+1}$. By compactness, it follows that L is covered by finitely many translates of K_{m+1}, say

$$L \subset \ell_1 K_{m+1} \cup \cdots \cup \ell_p K_{m+1}. \tag{1}$$

Now, for any $1 \leq i, j \leq p$, $\ell_i^{-1}\ell_j \in K_{n(i,j)}$ for some $n(i,j) \in \mathbb{Z}$. Set $N := \max\limits_{1 \leq i,j \leq p} n(i,j)$, so that $\ell_i^{-1}\ell_j \in K_N$ for any pair $1 \leq i, j \leq p$. Let $\ell, \ell' \in L$. From (1), there exist $i, j \in \{1, \dots, p\}$ with $\ell \in \ell_i K_{m+1}$, $\ell' \in \ell_j K_{m+1}$, so we may write $\ell = \ell_i z$, $\ell' = \ell_j z'$ for some $z, z' \in K_{m+1}$, and it follows that

$$\ell^{-1}\ell' = z^{-1}\ell_i^{-1}\ell_j z'.$$

Hence $d(\ell, \ell') \leq 2^{m+1} + 2^N + 2^{m+1} = 2^N + 2^{m+2}$, and this estimate is uniform over $\ell, \ell' \in L$. Therefore L has finite diameter with respect to d.

(ii) Let $g, h \in G$ and $\varepsilon > 0$. Let $n \geq 0$ be so that $2^{-n} \leq \frac{\varepsilon}{2}$. As K_{-n} is a neighborhood of e, $(g,h)(K_{-n} \times K_{-n})$ is a neighborhood of $(g,h) \in G \times G$, and for $(x,y) \in (g,h)(K_{-n} \times K_{-n})$, we have $x \in gK_{-n}$, $y \in hK_{-n}$, $g \in xK_{-n}$, $H \in yK_{-n}$. Then, by the triangle inequality and the left-invariance of d, one has

$$\begin{aligned} d(x,y) &\leq d(x,g) + d(g,h) + d(h,y) \\ &= d(e, x^{-1}g) + d(g,h) + d(e, h^{-1}y) \\ &\leq 2^{-n} + d(g,h) + 2^{-n} \\ &\leq d(g,h) + \varepsilon \end{aligned}$$

and similarly $d(g,h) \leq d(x,y) + \varepsilon$. Hence $|d(x,y) - d(g,h)| \leq \varepsilon$, and thus d is continuous at $(g,h) \in G \times G$.

(iv) Clearly, if $g \in G$ has $|g| = 0$, then $g \in K_n$ for any $n \in \mathbb{Z}$ by (iii). Conversely, if $g \in K_n$ for any $n \in \mathbb{Z}$, it directly follows from the definition of d that $|g| = d(e,g) < 2^n$ for any $n \in \mathbb{Z}$. Letting $n \to -\infty$ shows that $|g| = 0$, as claimed.

In particular, if $\bigcap\limits_{n \in \mathbb{Z}} K_n = \{e\}$ and that $d(g,h) = 0$ for some $g, h \in G$, then $d(e, g^{-1}h) = 0$ by left-invariance, so $g^{-1}h = e$, and thus $g = h$. Therefore d is a metric.

(iii) Explicitly, we must show that the following: let $w \in G$, $n \in \mathbb{Z}$, $k \geq 0$, $n_1, \dots, n_k \in \mathbb{Z}$, $v_1 \in K_{n_1}, \dots, v_k \in K_{n_k}$ with

$$w = v_1 \dots v_k, \; 2^{n_1} + \cdots + 2^{n_k} < 2^n.$$

Then $w \in K_n$.

If $k = 0$, then $w = e$ and we are done. If $k = 1$, then $2^{n_1} < 2^n$ implies $n_1 < n$, *i.e.* $n_1 \leq n-1$, so $w = v_1 \in K_{n_1} \subset K_{n-1} \subset K_n$, and we are done. If $k = 2$, then $2^{n_1} + 2^{n_2} < 2^n$ implies $n_1, n_2 < n$, *i.e.* $n_1, n_2 \leq n-1$, so $w = v_1 v_2 \in K_{n_1} K_{n_2} \subset K_{n-1}K_{n-1} \subset K_n$, so we are done in this case as well. We continue by induction on $k \geq 3$, assuming the claim is proved up to $k-1$. Hence suppose w can be written $w = v_1 \dots v_k$ with $v_1 \in K_{n_1}, \dots v_k \in K_{n_k}$ and that

$$2^{n_1} + \dots + 2^{n_k} < 2^n. \tag{2}$$

We now show the following claim:

Claim. There exists an index $j \in \{1, \dots, k\}$ so that $w = w_1 v_j w_2$ with now $w_1 = v_1 \dots v_{j-1}$, $w_2 = v_{j+1} \dots v_k$ satisfying

$$|w_1|, |w_2| < 2^{n-1}.$$

First of all, if there exists $j \in \{1, \dots, k\}$ with $n_j = n-1$, this index j is necessarily unique by the condition (2), so we can write $w = w_1 v_j w_2$, and

$$2^{n_1} + \dots + 2^{n_{j-1}} + 2^{n_{j+1}} + \dots + 2^{n_k} = 2^{n_1} + \dots + 2^{n_k} - 2^{n_j} < 2^n - 2^{n-1} = 2^{n-1}$$

using (2). A fortiori, $2^{n_1} + \dots + 2^{n_{j-1}} < 2^{n-1}$ and $2^{n_{j+1}} + \dots + 2^{n_k} < 2^{n-1}$, so that $|w_1|, |w_2| < 2^{n-1}$, and the claim is proved in this case.

Otherwise, $n_j \leq n-2$ for all $j \in \{1, \dots, k\}$, and we denote ℓ the largest index $i \in \{1, \dots, k\}$ for which $2^{n_1} + \dots + 2^{n_i} < 2^{n-1}$. If $\ell = k$, it is enough to take $j = 2$. Otherwise, if $\ell < k$, we set $j := \ell + 1$, so that $w = w_1 v_j w_2$ as above. On the one hand, $|w_1| \leq 2^{n_1} + \dots + 2^{n_{j-1}} = 2^{n_1} + \dots + 2^{n_\ell} < 2^{n-1}$ by definition of ℓ. On the other hand, we have

$$|w_2| \leq 2^{n_{j+1}} + \dots + 2^{n_k} = 2^{n_{\ell+2}} + \dots + 2^{n_k}$$

and the latter is strictly less than 2^{n-1} since the first part of the sum is $2^{n_1} + \dots + 2^{n_{\ell+1}} \geq 2^{n-1}$ by definition of ℓ and the total sum

$$2^{n_1} + \dots + 2^{n_{\ell+1}} + 2^{n_{\ell+2}} + \dots + 2^{n_k}$$

is strictly less than 2^n by assumption. This conludes the proof of the claim in this case.

Now, applying the induction hypothesis to w_1, w_2 in the claim shows that $w_1, w_2 \in K_{n-1}$, whence

$$w = w_1 v_j w_2 \in K_{n-1} K_{n_j} K_{n-1} \subset K_{n-1} K_{n-1} K_{n-1} \subset K_n.$$

This completes the inductive step, and also our proof of (iii). □

Our first metrisation criterion is usually referred to as the *Birkhoff-Kakutani theorem.*

Theorem 1.32. *Let G be a topological group. The following claims are equivalent.*

(i) *The group G is first-countable.*

(ii) *The group G is metrisable.*

(iii) *There exists a left-invariant compatible metric on* G.

Proof. To start, the implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are obvious. Hence we must only prove that (i) $\Longrightarrow$ (iii).

Assume that G is first-countable, and denote $(V_n)_{n\geq 1}$ a countable basis of neighborhoods of e. For $n \geq 0$, let $K_n := G$. For $n \geq 1$, choose inductively a symmetric neighborhood K_{-n} of e so that $K_{-n} \subset V_n$ and $K_{-n}K_{-n}K_{-n} \subset K_{-n+1}$. By construction, the sequence $(K_n)_{n\in\mathbb{Z}}$ satisfies the assumptions of Lemma 1.31, and we denote d the pseudo-metric provided by this lemma. First of all note that

$$\bigcap_{n\in\mathbb{Z}} K_n = \bigcap_{n\geq 1} K_{-n} \subset \bigcap_{n\geq 1} V_n = \{e\}$$

where the last equality follows from Proposition 1.22, that applies since G is Hausdorff. Thus, by Lemma 1.31(iv), d is a metric on G, which is furthermore left-invariant by (i) of the same lemma. Lastly, by point (iii) of Lemma 1.31, for any $n \in \mathbb{Z}$, the open ball with respect to d centered at $e \in G$ of radius 2^n is contained in K_n, and conversely from the definition of d, K_n is contained in the open ball with respect to d of radius 2^n centered at $e \in G$. This shows that the topology induced by d coincides with that of G, and thus that d is a left-invariant compatible metric on G. □

We can actually strengthen the conclusion if G is moreover locally compact.

Theorem 1.33. *Let G be a locally compact group. The following claims are equivalent.*

(i) *The group G is second-countable.*

(ii) *The group G is* $\sigma-$*compact and first-countable.*

(iii) *There exists a left-invariant proper compatible metric on* G.

Proof. By Theorem 1.5, implications (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (ii) are straightforward.

Assume now that (ii) holds, and let $(L_n)_{n\geq 0}$ be a sequence of symmetric compact subsets of G containing e so that

$$G = \bigcup_{n\geq 0} L_n.$$

Let also $(V_n)_{n\geq 0}$ be a countable basis of neighborhoods of e, with V_0 relatively compact. Define $K_0 := L_0 \cup \overline{V_0}$ and $K_{n+1} := K_nK_nK_n \cup L_{n+1}$ for any $n \geq 0$. For $n \geq 1$, choose inductively a symmetric neighborhood K_{-n} of e so that $K_{-n}K_{-n}K_{-n} \subset K_{-n+1}$, as the previous proof. By construction, the sequence $(K_n)_{n\in\mathbb{Z}}$ satisfies all assumptions in Lemma 1.31, so the pseudo-metric d provided by this result is once again a left-invariant continuous compatible metric on G, which is additionally proper thanks to Lemma 1.31. □

Theorem 1.34. *Let G be a* $\sigma-$*compact locally compact group.*

For any sequence $(U_n)_{n\geq 1}$ *of neighborhoods of* e *in* G*, there exists a normal compact subgroup* K *of* G *contained in* $\bigcap_{n\geq 1} U_n$ *so that* G/K *is metrisable.*

Proof. As G is σ−compact, we find a sequence $(L_n)_{n\geq 0}$ of compact subsets of G so that $e \in L_n \subset L_{n+1}$ for any $n \geq 0$ and

$$G = \bigcup_{n\geq 0} L_n.$$

For $n \geq 0$, let $K_n := G$ and for $n \geq 1$, we define inductively a symmetric compact neighborhood K_{-n} of e as follows. Suppose that $K_0, \ldots, K_{-n}$ have been defined. The map

$$\begin{aligned} L_n \times G &\longrightarrow G \\ (\lambda, k) &\longmapsto \lambda k \lambda^{-1} \end{aligned}$$

is continuous and identically equal to e on $L_n \times \{e\}$. Thus, for any $\lambda \in L_n$, there is an open neighborhood V_λ of λ and a compact neighborhood W_λ of e so that $\ell k \ell^{-1} \in K_{-n}$ for all $\ell \in V_\lambda$ and $k \in W_\lambda$. Then $L_n \subset \bigcup_{\lambda\in L_n} V_\lambda$, whence by compactness

$$L_n \subset \bigcup_{i=1}^{j} V_{\lambda_j}$$

for some finitely many $V_{\lambda_1}, \ldots, V_{\lambda_j}$. Set $K_{-n-1} := \bigcap_{i=1}^{j} W_{\lambda_j}$.

By construction, $\ell k \ell^{-1} \in K_{-n}$ for all $\ell \in L_n$ and $k \in K_{-n-1}$. Up to replacing K_{-n-1} by a smaller symmetric compact neighborhood of e, we can assume that $K_{-n-1} \subset U_n$ and $K_{-n-1}K_{-n-1}K_{-n-1} \subset K_{-n}$.

Let now $K := \bigcap_{n\geq 1} K_{-n}$. Then K is a closed subgroup of G contained in $\bigcap_{n\geq 1} U_n$. Let $g \in G$. There exists $n_0 \geq 1$ so that $g \in L_n$ for any $n \geq n_0$. Thus, for $n \geq n_0$, one gets

$$gK_{-n-1}g^{-1} \subset K_{-n} \subset K_{-n_0}$$

and thus $gKg^{-1} \subset K_{-n}$. It follows that $gKg^{-1} \subset \bigcap_{n\geq n_0} K_{-n} = K$, and thus K is normal. Applying now Lemma 1.31, we find a left-invariant proper continuous pseudo-metric d on G with the property that $d(e,g) = 0$ if and only if $g \in K$. This induces a left-invariant proper compatible metric on G/K. □

1.4 Compactly generated groups

Definition 1.35. Let G be a group. A generating set S for G is a subset $S \subset G$ so that, for any $g \in G$, there exist $n \geq 0$ and $s_1, \ldots, s_n \in S \cup S^{-1}$ so that

$$g = s_1 \ldots s_n.$$

A topological group G is compactly generated if it has a compact generating set S. In that case, we write $G = \langle S \rangle$.

Equivalently, a group G is generated by S if

$$G = \bigcup_{n \geq 0} \overline{S}^n.$$

where $\overline{S} := S \cup S^{-1} \cup \{e\}$.

Example 1.36. (i) Compact groups, such as S^1, $\mathrm{O}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, are compactly generated.

(ii) If G is connected and locally compact, then G is compactly generated, as a consequence of Corollary 1.26. For instance, $\mathrm{SL}_n(\mathbb{R})$ is compactly generated for any $n \geq 1$.

(iii) A locally compact group always has compactly generated open subgroups.

(iii) A discrete group is compactly generated if and only if it is finitely generated, *i.e.* it has a finite generating set. We will see plenty of examples below.

Here is a structural result on compactly generated groups.

Proposition 1.37. *Let $G = \langle S \rangle$ be a locally compact compactly generated group.*

(i) *For n large enough, $\overline{S}^n$ is a neighborhood of $e \in G$.*

(ii) *For any compact subset $K \subset G$, there is $k \geq 0$ so that K is contained in the interior of $\overline{S}^k$.*

(iii) *For every other compact generating set T of G, there exist $k, \ell \in \mathbb{N}$ so that $T \subset \overline{S}^k$ and $S \subset \overline{T}^\ell$.*

Proof. (i) As S generates G, we have

$$G = \bigcup_{n \geq 0} \overline{S}^n$$

and as G is a Baire group (since it is locally compact), there exists $m \in \mathbb{N}$ so that $\overline{S}^m$ has non-empty interior. Then the interior of $\overline{S}^n$ is an open neighborhood of e for any $n \geq 2m$.

(ii) Let $x \in K$. There exists $n_x \in \mathbb{N}$ with $x \in \overline{S}^{n_x}$, thus x is in the interior of $\overline{S}^{n_x+2m}$, where $m \in \mathbb{N}$ is as in (i). It follows that

$$K \subset \bigcup_{k \geq 0} (\overline{S}^k)^\circ$$

and as K is compact, the conclusion follows.

(iii) follows from (ii) applied to $K = T$ first and $K = S$ then. □

Definition 1.38. Let G be a topological group. A subgroup $H \leqslant G$ is cocompact if there exists a compact subset K of G so that $G = KH$.

In the sequel, we will also make use of the following technical lemma.

Lemma 1.39. *Let G be a locally compact topological group, H a closed subgroup, and $\pi \colon G \longrightarrow G/H$ the canonical projection.*

Then every compact subset of G/H is the image under π of a compact subset of G.

Proof. See for instance [5, lemma 2.C.9]. □

1.5 Finitely generated groups

Among topological groups, the class of discrete groups play a prominent role, and brings numerous examples whose behaviours are already delicate to analyse. The goal of this subsection is to introduce several classes of groups of interest. They will constitute some of our running examples that will follow us for the rest of this text.

We are in particular concerned by finitely generated groups, *i.e.* groups that have a finite generating set.

Example 1.40. (i) Finite groups are finitely generated.

(ii) The group $(\mathbb{Z},+)$ is finitely generated, and $S=\{-1,1\}$ is a finite generating set. More generally, if $p,q\in\mathbb{Z}$ are coprime, then $\{\pm p,\pm q\}$ is a symmetric generating set for $\mathbb{Z}$.

(iii) In fact, for any $d\geq 1$, the group $\mathbb{Z}^d$ is finitely generated, and a symmetric generating set is given by the "canonical" basis

$$\{\pm(1,0,\dots,0),\pm(0,1,0,\dots,0),\dots,\pm(0,0,\dots,0,1)\}.$$

(iv) If $d\geq 1$, the non-abelian free group F_d of rank d is finitely generated, a generating set being given by the equivalence classes of words of length one over a set S of cardinality d.

(v) It is not hard to check that the *Heisenberg group*

$$H(\mathbb{Z}) := \left\{\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} : a,b,c\in\mathbb{Z}\right\}$$

is generated by the three matrices

$$x=\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ y=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix},\ z=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

(vi) The group $(\mathbb{Q},+)$ is not finitely generated. Indeed, suppose for a contradiction that $\mathbb{Q}$ is generated by finitely many rationals $\frac{p_1}{q_1},\dots,\frac{p_n}{q_n}$. Any finite sum of these fractions or their inverses is a rational number with denominator at most $q_1\dots q_n$. Letting $N := q_1\dots q_n$, it follows that $\frac{1}{N+1}$ cannot be written using $\frac{p_1}{q_1},\dots,\frac{p_n}{q_n}$ or their inverses, a contradiction.

(vii) The group $\mathrm{SL}_2(\mathbb{Z})$ of determinant one 2×2 matrices with integer entries is finitely generated, for instance by

$$a=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},b=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

or even by two matrices of finite order (see *e.g.* [4, corollary 2.6]).

(viii) The group $D_\infty := \langle a,t : a^2=1, ata^{-1}=t^{-1}\rangle$ is called the *infinite dihedral group*, and generalizes finite dihedral groups allowing a rotation of infinite order. In fact, this group is isomorphic to

$$\langle a,b : a^2=b^2=1\rangle = \mathbb{Z}_2 * \mathbb{Z}_2$$

the free product of two cyclic groups of order 2.

Note that a finitely generated group is always countable, but the converse does not hold, as shown by Example 1.40(vi).

The following proposition shows that being finitely generated in stable under group extensions.

Proposition 1.41. *Let G be a group and $N \lhd G$.*

If G is finitely generated, then G/N is finitely generated. Conversely, if $N, G/N$ are finitely generated, then G is finitely generated.

In fact, similar statements are true for general topological groups. We treat here the discrete case for keeping the exposition relatively short, and we refer to [5, proposition 2.C.5] for the general case.

Proof. If G is finitely generated and $\pi\colon G \longrightarrow G/N$ is the natural surjection, then the image under π of a generating set for G is a generating set for G/N.

Conversely, let $\{g_1, \ldots, g_n\}$ be a generating set for N and $\{h_1N, \ldots, h_mN\}$ a generating set for G/N. Fix $g \in G$. Then there exists $\varepsilon_1, \ldots, \varepsilon_m \in \{-1, 1\}$ so that

$$gN = (h_1N)^{\varepsilon_1} \ldots (h_mN)^{\varepsilon_m} = (h_1^{\varepsilon_1} \ldots h_m^{\varepsilon_m})N$$

and it follows that $g(h_1^{\varepsilon_1} \ldots h_m^{\varepsilon_m})^{-1} \in N$. We can then write

$$g(h_1^{\varepsilon_1} \ldots h_m^{\varepsilon_m})^{-1} = g_1^{\delta_1} \ldots g_n^{\delta_n}$$

for some $\delta_1, \ldots, \delta_n \in \{-1, 1\}$, and thus $g = g_1^{\delta_1} \ldots g_n^{\delta_n} h_1^{\varepsilon_1} \ldots h_m^{\varepsilon_m}$. This proves that

$$\{g_1, \ldots, g_n, h_1, \ldots, h_m\}$$

is a finite generating set for G, and the proof is complete. $\square$

On the other hand, it is in general not true that subgroups of finitely generated groups are themselves finitely generated. To produce such an example, we introduce an additional group construction.

Definition 1.42. Let A, B be two groups. Their wreath product $A \wr B$ is the group defined by

$$\left(\bigoplus_B A\right) \rtimes B$$

where B acts on the direct sum by precomposition, *i.e.*

$$(b \cdot f)(b') := f(b^{-1}b')$$

for any $b, b' \in B$ and any $f \in \bigoplus_B A$.

Hence, elements of $A \wr B$ are pairs (f, b) where f is a finitely supported function on B (*i.e.* $f(b) = e_A$ for all but finitely many $b \in B$) and $b \in B$. The multiplication law is given by

$$(f, b)(f', b') = (f + b \cdot f', bb')$$

for all $f, f' \in \bigoplus_B A$, $b, b' \in B$, where "+" stands for the composition law in the direct sum.

We then prove that this construction preserves finite generation.

Proposition 1.43. *If A, B are finitely generated, then $A \wr B$ is finitely generated.*

Proof. Let $S_A = \{a_1, \dots, a_n\}$ be a generating set for A, and let $S_B = \{b_1, \dots, b_m\}$ be a generating set for B. For any $a \in A$, let $\delta_a \in \bigoplus_B A$ be defined by $\delta_a(e_B) = a$ and $\delta_a(b) = e_A$ for any $b \neq e_B$. Let also 1 denote the neutral element of the direct sum, defined as $1(b) = e_A$ for any $b \in B$. We claim that the finite set

$$\{(\delta_{a_i}, e_B), (1, b_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$$

is a generating set for $A \wr B$.

First, as the multiplication in $A \wr B$ is the multiplication of B in the second component, and as S_B generates B, it is enough to prove that any pair of the form (f, e_B) is a product of $(\delta_{a_1}, e_B), \dots, (\delta_{a_n}, e_B)$. Since f is finitely supported, it is enough to prove that any pair of the form (δ_a, e_B), $a \in A$, is a product of $(\delta_{a_1}, e_B), \dots, (\delta_{a_n}, e_B)$. For $a \in A$, write

$$a = a_{i_1} \dots a_{i_k}$$

for some $i_1, \dots, i_k \in \{1, \dots, n\}$, and then it follows that

$$(\delta_a, e_B) = (\delta_{a_{i_1}}, e_B) \dots (\delta_{a_{i_n}}, e_B).$$

Thus $A \wr B$ is finitely generated. $\square$

From this result, it follows that $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is finitely generated, but it contains $\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ as a subgroup, and the latter is not finitely generated.

However, there is a class of finitely generated groups having all their subgroups finitely generated.

Definition 1.44. A group G is called polycyclic if it has a sequence of subgroups

$$H_0 = \{e_G\} \leqslant H_1 \leqslant \dots \leqslant H_{s-1} \leqslant H_s = G$$

so that $H_i \lhd H_{i+1}$ and the quotient group H_{i+1}/H_i is cyclic, for any $i = 0, \dots, s-1$.

In particular, since cyclic groups are abelian, any polycyclic group is solvable.

Proposition 1.45. *Let G be a polycyclic group.*

Then any subgroup of G is finitely generated.

Proof. Let
$$H_0 = \{e_G\} \leqslant H_1 \leqslant \cdots \leqslant H_{s-1} \leqslant H_s = G$$
be a sequence as in Definition 1.44. Observe that H_i is polycyclic as well, for any $i = 0, \dots, s$. Let now H be a subgroup of G. We show that H is finitely generated by induction on s. If $s = 0$ there is nothing to prove. Assume that the statement is proved for any group with a sequence of subgroups as above of length at most $s-1$. Thus the subgroup $H \cap H_{s-1} \leqslant H_{s-1}$ is finitely generated. Also $G/H_{s-1} = H_s/H_{s-1}$ is either cyclic finite or isomorphic to $\mathbb{Z}$, and in both cases this quotient has all its subgroups finitely generated. Thus, from
$$H/(H \cap H_{s-1}) \cong HH_{s-1}/H_{s-1} \leqslant G/H_{s-1}$$
we deduce that $H/(H \cap H_{s-1})$ is finitely generated. Hence H is finitely generated as a consequence of Proposition 1.41. □

Example 1.46. As we proved above that $\mathbb{Z}_2 \wr \mathbb{Z}$ has a subgroup which is not finitely generated, it follows that $\mathbb{Z}_2 \wr \mathbb{Z}$ is not polycyclic. As it is nonetheless solvable, this shows that the class of polycyclic groups is strictly contained into the class of solvable groups.

As the class of solvable groups, polycyclic groups enjoy various stability properties with respect to basic group theoretic constructions.

Proposition 1.47. *Let G be a polycyclic group.*

Then its subgroups, quotients, homomorphic images, or extensions by polycyclic groups, are polycyclic groups.

Proof. We show the statement for homomorphic images, and the other ones are very similar. Assume G is polycyclic and let $f\colon G \longrightarrow H$ be a surjective group morphism. Taking a sequence
$$H_0 = \{e_G\} \leqslant H_1 \leqslant \cdots \leqslant H_{s-1} \leqslant H_s = G$$
of subgroups of G as in Definition 1.44 and pushing it through f provides a sequence
$$f(H_0) = \{e_H\} \leqslant f(H_1) \leqslant \cdots \leqslant f(H_{s-1}) \leqslant f(H_s) = H$$
of subgroups of H so that each subgroup is normal in the next one and all successive quotients are cyclic, since the image of a cyclic group through a group morphism is still a cyclic group. Thus H is polycyclic, as desired. □

Let us also recall the definition of nilpotent groups.

Definition 1.48. Let G be a group. The lower central series of G is the sequence $(\gamma_i(G))_{i\geq 1}$ of subgroups of G recursively defined by
$$\gamma_1(G) := G,\ \gamma_{i+1}(G) := [\gamma_i(G), G]\,,\ i \geq 1.$$

From this definition, an induction on $i \geq 1$ and the fact that automorphisms preserve commutators, one sees that $\gamma_i(G)$ is characteristic in G for any $i \geq 1$. In particular, $\gamma_i(G)$ is normal in G for any $i \geq 1$. Moreover, $\gamma_{i+1}(G) \subset \gamma_i(G)$ for any $i \geq 1$.

Definition 1.49. A group G is called nilpotent if its lower central series terminates, that is there is $i \geq 1$ so that $\gamma_i(G) = \{e_G\}$. In this case, the least integer $c \geq 1$ so that $\gamma_{c+1}(G) = \{e_G\}$ is called the nilpotency class of G.

Example 1.50. (i) Any abelian group is nilpotent, of nilpotency class $c = 1$.

(ii) In the Heisenberg group $H(\mathbb{Z})$, from Example 1.40(v), a direct computation shows that in fact $z = [x, y]$ and that $xz = zx$, $yz = zy$. In particular, z is central, thus so is any commutator, whence $[[H(\mathbb{Z}), H(\mathbb{Z})], H(\mathbb{Z})] = \{I_3\}$. This shows that $H(\mathbb{Z})$ is nilpotent of nilpotency class $c = 2$. More generally, the group of upper unitriangular $n \times n$ matrices over a unital commutative ring is nilpotent of nilpotency class $n - 1$.

We can now prove that any finitely generated nilpotent group is an example of a polycyclic group.

Proposition 1.51. *Let G be a finitely generated nilpotent group.*

Then there exists a sequence of subgroups

$$N_{s+1} = \{e_G\} \leqslant N_s \leqslant \cdots \leqslant N_2 \leq N_1 = G$$

so that $N_{i+1} \lhd N_i$ and N_i/N_{i+1} is cyclic, for any $i = 1, \ldots, s$. In particular, G is polycyclic.

Proof. Let $(\gamma_i(G))_{i\geq 1}$ be the lower central series of G. Since $\gamma_i(G)/\gamma_{i+1}(G)$ is finitely generated and abelian (see [3, proposition 2.28]), by the structure theorem of finitely generated abelian groups, there is a sequence of subgroups

$$\gamma_{i+1} = N_{i,t_i} \leqslant N_{i,t_i-1} \leqslant \cdots \leqslant N_{i,2} \leqslant N_{i,1} = \gamma_i$$

with $N_{i,j+1} \lhd N_{i,j}$ and $N_{i,j}/N_{i,j+1}$ cyclic for any $j = 1, \ldots, t_i - 1$. We rename the sequence $N_{1,1}, N_{1,2}, \ldots, N_{1,t_1} = N_{2,1}, \ldots, N_{2,t_2} = N_{3,1}, \ldots$ as $N_1, N_2, N_3, \ldots$ to conclude. $\square$

Combining the latter statement and Proposition 1.45, we deduce that any subgroup of a finitely generated nilpotent group is finitely generated.

We conclude this section mentioning two important structural results on nilpotent and polycyclic groups. The first one is a sort of converse to Example 1.50(ii). For the statement, recall that a group G is *linear* if it is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{K})$, for some $n \geq 1$ and some field $\mathbb{K}$.

Theorem 1.52. *Any finitely generated nilpotent group is linear. In fact, such a group is embeddable into $\mathrm{GL}_n(\mathbb{Z})$ for some $n \geq 1$.*

The second statement is due to Malcev, and furnishes important examples of polycyclic groups.

Theorem 1.53. *Let G be a finitely generated and solvable subgroup of $\mathrm{GL}_n(\mathbb{Z})$.*

Then G is polycyclic.

1.6 Haar measures on locally compact groups

We close this chapter recalling elementary facts on Haar measures for locally compact topological groups.

Let thus G be a locally compact group. Its *Borel $\sigma-$algebra $\mathcal{B}$* is the $\sigma-$algebra generated by the open subsets of G. A measure μ defined on $\mathcal{B}$ is *left-invariant* if $\mu(gE) = \mu(E)$ for any $g \in G$, $E \in \mathcal{B}$, and it is called *regular* if

(i) For all $B \in \mathcal{B}$, $\mu(B) = \inf_{B\subset V,\, V \text{ open}} \mu(V)$.

(ii) For all open subset $U \subset G$, $\mu(U) = \sup_{K\subset U,\, K \text{ compact}} \mu(K)$.

(iii) For all compact subset $K \subset G$, $\mu(K) < \infty$.

Theorem 1.54. *Let G be a locally compact group, and let $\mathcal{B}$ be its Borel $\sigma-$algebra.*

Then there exists a regular left-invariant measure μ on $\mathcal{B}$. Such a measure is unique up to multiplication by positive constants.

Such a measure is then usually referred to as a *Haar measure* on the group.

Let us mention two basic properties of Haar measures. Recall that the *support* of a measure μ defined on a group G is the smallest closed subset $F \subset G$ so that $\mu(G \setminus F) = 0$.

Proposition 1.55. *Let G be a locally compact group, and μ be a Haar measure on G. The following assertions hold.*

(i) $\operatorname{supp}(\mu) = G$.

(ii) *The group G is compact if and only if* $\mu(G) < \infty$.

Proof. We prove (i) by contradiction. Assume there is a non-empty open subset $U \subset G$ with $\mu(U) = 0$. Then $\mu(gU) = 0$ for any $g \in G$ by left-invariance. If now $K \subset G$ is compact, we may find finitely many group elements $g_1, \dots, g_n \in G$ so that $K \subset g_1U \cup \cdots \cup g_nU$, whence $\mu(K) = 0$ as well. Since μ is regular, it follows that $\mu = 0$, a contradiction. Thus $\mu(U) > 0$ for any open subset $U \subset G$, whence $\operatorname{supp}(\mu) = G$.

We now turn to the proof of (ii). Clearly if G is compact then $\mu(G) < \infty$ by regularity. Conversely, assume that G is not compact, and let U be a compact neighborhood of $e \in G$. By induction, we construct a sequence $(g_n)_{n\geq 1}$ so that

$$g_{n+1} \notin \bigcup_{i=1}^{n} g_iU$$

for any $n \in \mathbb{N}$. Appealing Proposition 1.18, choose a neighborhood V of e so that $V^{-1} = V$ and $V^{-1}V = V^2 \subset U$. Then $g_nV \cap g_mV = \emptyset$ if $n \neq m$, and thus

$$\mu(G) \geq \mu(\bigcup_{i\geq 1} g_iV) = \sum_{i=1}^{\infty} \mu(g_iV) = \sum_{i=1}^{\infty} \mu(V) = \infty$$

whence $\mu(G) = \infty$ as claimed. □

2. The metric coarse category

The goal of this chapter is to develop the appropriate framework to study topological groups as metric spaces.

2.1 Coarsely Lipschitz maps and large-scale Lipschitz maps

An *upper control* is a non-decreasing function $\Phi_+ : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, and a *lower control* is a non-decreasing function $\Phi_- : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ so that $\lim\limits_{t\to\infty} \Phi_-(t) = \infty$.

If X, Y are pseudo-metric spaces and $f : X \longrightarrow Y$ is a map, an *upper control for f* is an upper control Φ_+ so that

$$d_Y(f(x), f(x')) \leq \Phi_+(d_X(x, x'))$$

for any $x, x' \in X$, and dually a *lower control for f* is a lower control Φ_- so that

$$\Phi_-(d_X(x, x')) \leq d_Y(f(x), f(x'))$$

for any $x, x' \in X$.

Definition 2.1. Let X, Y be pseudo-metric spaces and let $f : X \longrightarrow Y$ be a map. We say that f is

(i) coarsely Lipschitz if there exists an upper control for f.

(ii) coarsely expansive if there exists a lower control for f.

(iii) a coarse embedding if it is coarsely Lipschitz and coarsely expansive.

(iv) essentially surjective if $f(X)$ is co-bounded in Y.

(v) a metric coarse equivalence if it is an essentially surjective coarse embedding.

If there exists a metric coarse equivalence $f : X \longrightarrow Y$, we say that X and Y are *coarsely equivalent.*

Let us start with equivalent reformulations of the above conditions.

Proposition 2.2. *Let X, Y be pseudo-metric spaces and $f : X \longrightarrow Y$ a map. The following are equivalent.*

(i) *The map f is coarsely Lipschitz.*

(ii) *For all $R \geq 0$, there exists $S \geq 0$ so that if $x, x' \in X$ have $d_X(x, x') \leq R$, then $d_Y(f(x), f(x')) \leq S$.*

(iii) *For any sequence of points $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}$ in X with $\sup\limits_{n\in\mathbb{N}} d_X(x_n, x'_n) < \infty$, we have*

$$\sup_{n\in\mathbb{N}} d_Y(f(x_n), f(x'_n)) < \infty.$$

Proof. (i) $\Longrightarrow$ (ii) : Suppose that f is coarsely Lipschitz, and denote Φ_+ an upper control for f. Let $R \geq 0$, and set $S := \Phi_+(R) \geq 0$. Then, if $x, x' \in X$ are so that $d_X(x, x') \leq R$, it follows that

$$d_Y(f(x), f(x')) \leq \Phi_+(d_X(x, x')) \leq \Phi_+(R) = S$$

since f is coarsely Lipschitz and Φ_+ is non-decreasing. Thus (ii) holds.

(ii) $\Longrightarrow$ (iii) : Let $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}$ be two sequences of points in X with $C < \infty$, where $C := \sup_{n\in\mathbb{N}} d_X(x_n, x'_n)$. Using (ii), there is $S \geq 0$ so that

$$d_Y(f(x), f(x')) \leq S$$

if $d_X(x, x') \leq C$. As $d_X(x_n, x'_n) \leq C$ for any $n \in \mathbb{N}$, we have also

$$d_Y(f(x_n), f(x'_n)) \leq S$$

for any $n \in \mathbb{N}$, and thus $\sup_{n\in\mathbb{N}} d_Y(f(x_n), f(x'_n)) \leq S < \infty$, which shows (iii).

(iii) $\Longrightarrow$ (i) : For $c \in \mathbb{R}_+$, define

$$\Phi_+(c) := \sup\{d_Y(f(x), f(x')) : x, x' \in X, d_X(x, x') \leq c\}.$$

Then Φ_+ is positive and non-decreasing. Towards a contradiction, suppose that $\Phi_+(c) = \infty$ for some $c \in \mathbb{R}_+$. This implies there exist two sequences $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}} \subset X$ with $d_X(x_n, x'_n) \leq c$ for any $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} d_Y(f(x_n, f(x'_n)) = \Phi_+(c) = \infty$$

which is excluded by (iii). Hence Φ_+ takes only finite values, and thus is indeed an upper control for f. □

Dualising the above proof, one gets the same statement for coarsely expansive maps.

Proposition 2.3. *Let X, Y be pseudo-metric spaces and $f\colon X \longrightarrow Y$ a map. The following are equivalent.*

(i) *The map f is coarsely expansive.*

(ii) *For all $r \geq 0$, there exists $s \geq 0$ so that if $x, x' \in X$ have $d_X(x, x') \geq r$, then one has $d_Y(f(x), f(x')) \geq s$.*

(iii) *For any sequence of points $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}$ in X with $\lim_{n\to\infty} d_X(x_n, x'_n) = \infty$, we have*

$$\lim_{n\to\infty} d_Y(f(x_n), f(x'_n)) = \infty.$$

Given two maps $f, f'\colon X \longrightarrow Y$ between pseudo-metric spaces, we say that f' is *close* to f (or f' is *at bounded distance* from f) if there exists $C > 0$ so that

$$d_Y(f(x), f(x')) \leq C$$

for any $x, x' \in X$. This is the same as requiring that

$$\sup_{x\in X} d_Y(f(x), f(x')) < \infty$$

and, in this case, we write $f \sim f'$.

Lemma 2.4. *Closeness is an equivalence relation.*

Proof. Clearly $f \sim f$ as $d_Y(f(x), f(x)) = 0$ for any $x \in X$. Symmetry of $\sim$ follows from symmetry of d_Y, and transitivity follows from the triangle inequality for d_Y. □

The next result shows that properties from Definition 2.1 are invariant when taking close maps.

Proposition 2.5. *Let X, Y, Z be pseudo-metric spaces, $f, f' \colon X \longrightarrow Y$ two close maps, and $g, g' \colon Y \longrightarrow Z$ two close maps.*

(i) *The map f is coarsely Lipschitz (resp. coarsely expansive, a coarse embedding, essentially surjective, a metric coarse equivalence) if and only if f' is coarsely Lipschitz (resp. coarsely expansive, a coarse embedding, essentially surjective, a metric coarse equivalence).*

(ii) *If f, g are both coarsely Lipschitz (resp. coarsely expansive, coarse embeddings, essentially surjective, metric coarse equivalences), then $g \circ f$ is coarsely Lipschitz (resp. coarsely expansive, a coarse embedding, essentially surjective, a metric coarse equivalence).*

(iii) *If g is coarsely Lipschitz, then $g \circ f$ and $g' \circ f'$ are close.*

Proof. (i) Suppose f is coarsely Lipschitz, and let $C > 0$ be so that $d_Y(f(x), f'(x)) \leq C$ for any $x, x' \in X$. Let $R \geq 0$. As f is coarsely Lipschitz, we find $K \geq 0$ so that

$$d_X(x, x') \leq R \Longrightarrow d_Y(f(x), f(x')) \leq K.$$

Set $S := K + 2C$, and let $x, x' \in X$ with $d_X(x, x') \leq R$. Then it follows that

$$\begin{aligned} d_Y(f'(x), f'(x')) &\leq d_Y(f'(x), f(x)) + d_Y(f(x), f(x')) + d_Y(f(x'), f'(x')) \\ &\leq K + 2C \\ &= S. \end{aligned}$$

As $R \geq 0$ was arbitrary, Proposition 2.2 guarantees that f' is coarsely Lipschitz as well. The converse follows swapping the roles of f and f'.

Now, suppose that f is essentially surjective. As above, let $C > 0$ be so that

$$d_Y(f(x), f'(x)) \leq C$$

for any $x, x' \in X$, and let $C' > 0$ be so that any point in Y is at distance at most C' from the image of f. Let $y \in Y$, and choose $x \in X$ with $d_Y(f(x), y) \leq C'$. Then we get that

$$d_Y(f'(x), y) \leq d_Y(f'(x), f(x)) + d_Y(f(x), y) \leq C + C'.$$

Hence any point of Y is at distance at most $C + C'$ from the image of f', *i.e.* f' is essentially surjective. Once again, the converse follows by symmetry, and the proofs for the other properties are completely similar.

(ii) Here also we only do the proof for one of the properties, and similar arguments apply for the others. Suppose for instance that f and g are both coarsely expansive. Let $r \geq 0$. Applying Proposition 2.3(ii) to f, we find $s \geq 0$ so that

$$d_X(x, x') \geq r \Longrightarrow d_Y(f(x), f(x')) \geq s$$

and applying now Proposition 2.3(ii) to g, there is $t \geq 0$ so that

$$d_Y(y, y') \geq s \Longrightarrow d_Z(g(y), g(y')) \geq t.$$

Combining these two implications, we now conclude that if $x, x' \in X$ are so that $d_X(x, x') \geq r$, then $d_Z((g \circ f)(x), (g \circ f)(x')) \geq t$, proving that $g \circ f$ is coarsely expansive.

(iii) Let $C > 0$ be so that $d_Y(f(x), f'(x)) \leq C$ for any $x \in X$, and let $C' > 0$ playing the same role for g and g'. As g is coarsely Lipschitz, there is $K \geq 0$ so that $d_Z(g(y), g(y')) \leq K$ if $d_Y(y, y') \leq C$. Then for any $x \in X$ one has

$$d_Z(g(f(x)), g'(f'(x))) \leq d_Z(g(f(x)), g(f'(x))) + d_Z(g(f'(x)), g'(f'(x))) \leq K + C'$$

whence $g \circ f$ and $g' \circ f'$ are close. □

This proposition motivates then the next definition.

Definition 2.6. Let X, Y be pseudo-metric spaces. A coarse morphism from X to Y is a closeness class of coarsely Lipschitz maps from X to Y.

The metric coarse category is the category whose objects are pseudo-metric spaces and whose morphisms are coarse morphisms.

Definition 2.7. Let X, Y be pseudo-metric spaces and $f\colon X \longrightarrow Y$ be a map. We say that f is

(i) large-scale Lipschitz if it has an affine upper control, *i.e.* there exist $c_+ > 0$, $c'_+ \geq 0$ so that
$$d_Y(f(x), f(x')) \leq c_+ d_X(x, x') + c'_+$$
for any $x, x' \in X$.

(ii) large-scale expansive if it has an affine lower control, *i.e.* there exist $c_- > 0$, $c'_- \geq 0$ so that
$$d_Y(f(x), f(x')) \geq c_- d_X(x, x') - c'_-$$
for any $x, x' \in X$.

(iii) a quasi-isometric embedding if it is large-scale Lipschitz and large-scale expansive.

(iv) a quasi-isometry if it is an essentially surjective quasi-isometric embedding.

If there is a quasi-isometry $f\colon X \longrightarrow Y$, we say that X and Y are *quasi-isometric*, and we denote $X \sim_{Q.I.} Y$. As we will see below, $\sim_{Q.I.}$ is an equivalence relation among pseudo-metric spaces.

Remark 2.8. In particular, any large-scale Lipschitz map is coarsely Lipschitz, and any large-scale expansive map is coarsely expansive.

Example 2.9. (i) Consider $X = \mathbb{Z}$ with its usual distance (induced from that of $\mathbb{R}$) and $Y = \mathbb{R}$ with its usual distance. The natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry, since it is an isometric map and since any real number is at distance at most 1 from an integer, namely its integer part.

More generally, for any $n \geq 1$, the natural inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is a quasi-isometry, since it is an isometry and since any $n-$tuple of real numbers $(x_1, \ldots, x_n)$ is at distance at most $\sqrt{n}$ from an $n-$tuple of integers, namely $(\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)$.

(ii) Likewise, the natural inclusion $2\mathbb{Z} \hookrightarrow \mathbb{Z}$ is also a quasi-isometry, since it is an isometric map and since any integer is at distance at most 1 from an even integer.

Here goes the natural analog of Proposition 2.5 for large-scale Lipschitz/expansive maps.

Proposition 2.10. *Let X, Y, Z be pseudo-metric spaces, $f, f' \colon X \longrightarrow Y$ two close maps, and $g, g' \colon Y \longrightarrow Z$ two close maps.*

(i) *The map f is large-scale Lipschitz (resp. large-scale expansive, a quasi-isometric embedding, a quasi-isometry) if and only if f' is large-scale Lipschitz (resp. large-scale expansive, a quasi-isometric embedding, a quasi-isometry).*

(ii) *If f, g are both large-scale Lipschitz (resp. large-scale expansive, quasi-isometric embeddings, quasi-isometries), then $g \circ f$ is large-scale Lipschitz (resp. large-scale expansive, a quasi-isometric embedding, a quasi-isometry).*

This in turn leads to a natural analog of the metric coarse category for large-scale Lipschitz maps.

Definition 2.11. Let X, Y be pseudo-metric spaces. A large-scale morphism from X to Y is a closeness class of large-scale Lipschitz maps from X to Y.

The large-scale category is the subcategory of the metric coarse category whose objects are pseudo-metric spaces and whose morphisms are large-scale Lipschitz morphisms.

Definition 2.12. Let X, Y be pseudo-metric spaces and $f \colon X \longrightarrow Y$ be a map. We say that f is

(i) Lipschitz if there is $c_+ > 0$ so that

$$d_Y(f(x), f(x')) \leq c_+ d_X(x, x')$$

for any $x, x' \in X$.

(ii) bilipschitz if there exist $c_+ > 0$, $c_- \geq 0$ so that

$$c_- d_X(x, x') \leq d_Y(f(x), f(x')) \leq c_+ d_X(x, x')$$

for any $x, x' \in X$.

(iii) a bilipschitz equivalence if it is bilipschitz and surjective.

Let us now give additional examples of such maps.

Example 2.13. (i) Let $f: X \longrightarrow Y$ be a map between two pseudo-metric spaces. If X has finite diameter, then f is large-scale expansive, since

$$d_Y(f(x), f(x')) \geq 0 \geq d_X(x, x') - \mathrm{diam}(X)$$

for all $x, x' \in X$. If rather $f(X)$ has finite diameter, then f is large-scale Lipschitz, since

$$d_Y(f(x), f(x')) \leq \mathrm{diam}(f(X))$$

for all $x, x' \in X$. Lastly, if Y has finite diameter and $X \neq \emptyset$, then f is essentially surjective. Combining these three facts, it follows that any non-empty pseudo-metric space of finite diameter is quasi-isometric to the one point space.

(ii) For any $p \geq 1$, the map $\mathrm{Id}_{\mathbb{R}^n}: (\mathbb{R}^n, d_\infty) \longrightarrow (\mathbb{R}^n, d_p)$ is a bilipschitz equivalence, since

$$d_\infty(x, y) \leq d_p(x, y) \leq n^{\frac{1}{p}} d_\infty(x, y)$$

for any $x, y \in \mathbb{R}^n$, where the metric d_p, d_∞ are defined as

$$d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^{\frac{1}{p}} \right)^p, \ d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|$$

for any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(iii) Let (X, d) be a pseudo-metric space. Define a metric d_1 by setting

$$d_1(x, x') := \max(1, d(x, x'))$$

for all $x \neq x' \in X$ and $d_1(x, x) = 0$ for all $x \in X$. Then the map $\mathrm{Id}_X: (X, d) \longrightarrow (X, d_1)$ is a quasi-isometry. Define now another metric $d_{\ln}$ on X by the formula

$$d_{\ln}(x, x') := \ln(1 + d(x, x')), \ x, x' \in X.$$

The map $\mathrm{Id}_X: (X, d) \longrightarrow (X, d_{\ln})$ is a metric coarse equivalence, since it is surjective and the functions $\Phi_-(t) = \Phi_+(t) = \ln(1+t)$ are lower and upper controls for Id_X. We claim that $\mathrm{Id}_X: (X, d) \longrightarrow (X, d_{\ln})$ is large-scale expansive if and only if (X, d) has finite diameter.

Proof. If (X, d) has finite diameter, Id_X is large-scale expansive by (i) above. Conversely, assume there exist $c_- > 0$, $c'_- \geq 0$ with

$$d_{\ln}(x, x') = \ln(1 + d(x, x')) \geq c_- d(x, x') - c'_-$$

for all $x, x' \in X$. Towards a contradiction, assume that $\mathrm{diam}(X, d) = \infty$, and pick two sequences $(x_n)_{n\in\mathbb{N}}$, $(x'_n)_{n\in\mathbb{N}}$ in X so that $d(x_n, x'_n) \to \infty$ as $n \to \infty$. It follows from the assumption that

$$\frac{\ln(1 + d(x_n, x'_n))}{d(x_n, x'_n)} \geq c_- - \frac{c'_-}{d(x_n, x'_n)}$$

for all $n \in \mathbb{N}$ large enough. Letting $n \to \infty$ in this inequality provides $c_- \leq 0$, a contradiction. Thus (X, d) must have finite diameter. □

(iv) A pseudo-metric space X is *hyperdiscrete* if the set

$$\{(x,x') \in X^2 : d_X(x,x') \le c, x \ne x'\}$$

is finite for any $c > 0$. Then every map from a hyperdiscrete pseudo-metric space to any pseudo-metric space is coarsely Lipschitz.

(v) Let X be a metric space, Y a pseudo-metric space, and suppose there is $c > 0$ so that $d_X(x,x') \ge c$ for all $x \ne x' \in X$. Then a map $f\colon X \longrightarrow Y$ is large-scale Lipschitz if and only if it is Lipschitz. Indeed, suppose that f is large-scale Lipschitz, which means there exist $c_+ > 0$, $c'_+ \ge 0$ so that

$$d_Y(f(x), f(x')) \le c_+ d_X(x,x') + c'_+$$

for any $x, x' \in X$. Now $1 \le \frac{d_X(x,x')}{c}$ for all $x, x' \in X$, and it follows that

$$d_Y(f(x), f(x')) \le \big(c_+ + \frac{c'_+}{c}\big) d_X(x,x')$$

for any $x, x' \in X$.

The next lemma ensures that control functions are almost invertible.

Lemma 2.14. (i) *Let* $\Phi_+\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be an upper control. The function* $\Psi_-\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ *defined by*

$$\Psi_-(s) := \inf\{r \ge 0 : \Phi_+(r) \ge s\},\ s \ge 0$$

is a lower control so that $\Psi_-(\Phi_+(t)) \le t$ *for all* $t \ge 0$.

(ii) *Let* $\Phi_-\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ *be an upper control. The function* $\Psi_+\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *defined by*

$$\Psi_+(s) := \sup\{r \ge 0 : \Phi_-(r) \le s\},\ s \ge 0$$

is an upper control so that $\Psi_+(\Phi_-(t)) \ge t$ *for all* $t \ge 0$.

Proof. We only prove (i) since the proof of (ii) is identical. Clearly $\Psi_-(s) \ge 0$ for any $s \ge 0$. Next, if $s_1 \le s_2$, and if $r \ge 0$ is so that $\Phi_+(r) \ge s_2$, then also $\Phi_+(r) \ge s_1$, whence $\Psi_-(s_1) \le r$. It follows that $\Psi_-(s_1) \le \inf\{r \ge 0 : \Phi_+(r) \ge s_2\} = \Psi_-(s_2)$, and Ψ_- is non-decreasing. Lastly, if $t \ge 0$, we have

$$\Psi_-(\Phi_+(t)) = \inf\{r \ge 0 : \Phi_+(r) \ge \Phi_+(t)\} \le t$$

since $t \in \{r \ge 0 : \Phi_+(r) \ge \Phi_+(t)\}$. □

Proposition 2.15. *Let* X, Y *be pseudo-metric spaces,* $f\colon X \longrightarrow Y$ *a coarsely Lipschitz map, and* $\overline{f}$ *the corresponding morphism in the metric coarse category. The following claims hold.*

(i) *If* $X \ne \emptyset$, $\overline{f}$ *is an epimorphism if and only if* f *is essentially surjective.*

(ii) *The morphism* $\overline{f}$ *is a monomorphism if and only if* f *is coarsely expansive.*

(iii) *The morphism $\overline{f}$ is an isomorphism if and only if f is a metric coarse equivalence. Moreover, if $X \neq \emptyset$, this holds if and only if $\overline{f}$ is an epimorphism and a monomorphism.*

Proof. (i) Assume first that f is essentially surjective, and let $c := \sup_{y \in Y} d_X(y, f(X))$. Consider a pseudo-metric space Z and two coarsely Lipschitz maps $h_1, h_2 \colon Y \longrightarrow Z$ so that $h_1 \circ f \sim h_2 \circ f$. Hence there is $c' > 0$ so that

$$d_Z(h_1(f(x)), h_2(f(x))) \leq c'$$

for any $x \in X$. Moreover, as h_1, h_2 are coarsely Lipschitz, we can find $c_1, c_2 > 0$ so that

$$d_Y(y, y') \leq c \Longrightarrow d_Z(h_1(y), h_1(y')) \leq c_1, \; d_Y(y, y') \leq c \Longrightarrow d_Z(h_2(y), h_2(y')) \leq c_2$$

for all $y, y' \in Y$. Let now $y \in Y$, and choose $x \in X$ with $d_Y(y, f(x)) \leq c$. It then follows from the above implications that

$$\begin{aligned} d_Z(h_1(y), h_2(y)) &\leq d_Z(h_1(y), h_1(f(x))) + d_Z(h_1(f(x)), h_2(f(x))) + d_Z(h_2(f(x)), h_2(y)) \\ &\leq c_1 + c' + c_2. \end{aligned}$$

As $y \in Y$ was arbitrary, this proves that $h_1 \sim h_2$, so $\overline{f}$ is an epimorphism.

Conversely, suppose f is not essentially surjective. Define $h_1, h_2 \colon Y \longrightarrow \mathbb{R}_+$ by $h_1(y) = 0$, and $h_2(y) = d_Y(y, f(X))$. Then $h_1 \circ f = h_2 \circ f = 0$, so $h_1 \circ f \sim h_2 \circ f$, but $h_1 \nsim h_2$, as $h_1(Y) = \{0\}$ is bounded in $\mathbb{R}_+$ while $h_2(Y)$ is not. As h_1, h_2 are coarsely Lipschitz (and thus morphisms in the metric coarse category), we deduce that $\overline{f}$ is not an epimorphism.

(ii) Suppose now that f is coarsely expansive. Let Φ_- be a lower control for f and Ψ_+ an upper control as in Lemma 2.14. Let W be a pseudo-metric space and let $h_1, h_2 \colon W \longrightarrow X$ be two coarsely Lipschitz maps so that $f \circ h_1 \sim f \circ h_2$. Hence there is $c > 0$ so that

$$d_Y(f(h_1(w)), f(h_2(w))) \leq c$$

for any $w \in W$. It follows that

$$\begin{aligned} d_X(h_1(w), h_2(w)) &\leq \Psi_+(\Phi_-(d_X(h_1(w), h_2(w)))) \\ &\leq \Psi_+(d_Y(f(h_1(w)), f(h_2(w)))) \\ &\leq \Psi_+(c) \end{aligned}$$

for any $w \in W$, which shows that $h_1 \sim h_2$. Thus $\overline{f}$ is a monomorphism.

Conversely, suppose that f is not coarsely expansive. This implies there exist $c > 0$ and $(x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}} \subset X$ with $\lim_{n \to \infty} d_X(x_n, x'_n) = \infty$ and

$$d_Y(f(x_n), f(x'_n)) \leq c$$

for all $n \in \mathbb{N}$. Consider now $W := \{n^2 : n \in \mathbb{N}\}$ endowed with the usual metric ($d_W(n^2, m^2) = |n^2 - m^2|, n, m \in \mathbb{N}$), and the maps $h_1, h_2 \colon W \longrightarrow X$, $h_1(n^2) = x_n$, $h_2(n^2) = x'_n$. As W is a hyperdiscrete pseudo-metric space, Example 2.13(iv) applies and guarantees that h_1, h_2 are coarsely Lipschitz. Now $f \circ h_1 \sim f \circ h_2$, but $h_1 \not\sim h_2$ as $d_X(x_n, x'_n) \to \infty$ when $n \to \infty$. Hence $\overline{f}$ is not a monomorphism.

(iii) Suppose that f is a metric coarse equivalence. If $X = Y = \emptyset$, there is nothing to prove. If $Y \neq \emptyset$, then so is X (since the only map $\emptyset \longrightarrow Y$ is not essentially surjective). Let Φ_-, Φ_+ be lower and upper controls for f and let $c > 0$ be so that $d_Y(y, f(X)) \leq c$ for any $y \in Y$. Let Ψ_+ be as in Lemma 2.14 and let Ψ_- be the lower control given by

$$\Psi_-(s) = \inf\{r \geq 0 : \Phi_+(r) + 2c \geq s\}, \ s \geq 0.$$

For each $y \in Y$, pick $x_y \in X$ so that $d_Y(y, f(x_y)) \leq c$, and define $g\colon Y \longrightarrow X$ by $g(y) := x_y$. Let $y, y' \in Y$. Then we have

$$\begin{aligned} d_X(g(y), g(y')) &\leq \Psi_+(\Phi_-(d_X(g(y), g(y')))) \\ &\leq \Psi_+(d_Y(f(g(y)), f(g(y')))) \\ &= \Psi_+(d_Y(f(x_y), f(x_{y'}))) \\ &\leq \Psi_+(d_Y(y, y') + 2c) \end{aligned}$$

which proves that $s \longmapsto \Phi_+(s + 2c)$ is an upper control for g, which is then coarsely Lipschitz. On the other hand, we have

$$\begin{aligned} d_Y(y, y') &\leq d_Y(y, f(g(y))) + d_Y(f(g(y)), f(g(y'))) + d_Y(f(g(y')), y') \\ &\leq \Phi_+(d_X(g(y), g(y')) + 2c \\ &= \widetilde{\Phi_+}(d_X(g(y), g(y'))) \end{aligned}$$

for any $y, y' \in Y$, where $\widetilde{\Phi_+}(s) := \Phi_+(s) + 2c$, $s \geq 0$. It follows that

$$\Psi_-(d_Y(y, y')) \leq \Psi_-(\widetilde{\Phi_+}(d_X(g(y), g(y')))) \leq d_Y(g(y), g(y'))$$

for all $y, y' \in Y$. Therefore g is coarsely expansive as well. By construction, we have $f \circ g \sim \mathrm{Id}_Y$, so $f \circ g \circ f \sim f$, and as $\overline{f}$ is a monomorphism by (ii), we conclude that $g \circ f \sim \mathrm{Id}_X$, and finally that $\overline{f}$ is an isomorphism with inverse $\overline{g}$.

Conversely, if $\overline{f}$ is an isomorphism, then f is essentially surjective by (i) and coarsely expansive by (ii), thus it is a metric coarse equivalence. □

It follows from this result and the fact that compositions of metric coarse equivalences are metric coarse equivalences that being coarsely equivalent is an equivalence relation among pseudo-metric spaces.

Definition 2.16. Let X, Y be pseudo-metric spaces and let $f\colon X \longrightarrow Y$ be a coarsely Lipschitz map. We say that f is

(i) coarsely right-invertible if there exists a coarsely Lipschitz map $g\colon Y \longrightarrow X$ so that $f \circ g \sim \mathrm{Id}_Y$.

(ii) coarsely left-invertible if there exists a coarsely Lipschitz map $g\colon Y \longrightarrow X$ so that $g \circ f \sim \mathrm{Id}_X$.

(iii) coarsely invertible if it is both left-invertible and right-invertible.

Remark 2.17. If $f\colon X \longrightarrow Y$ is coarsely Lipschitz and coarsely right-invertible, then $\overline{f}$ is an epimorphism. Dually, if f is coarsely left-invertible, then f is a monomorphism. In particular, f is coarsely invertible if and only if $\overline{f}$ is an isomorphism, *i.e.* if and only if f is a metric coarse equivalence.

Definition 2.18. Let Y be a pseudo-metric space. A subspace Z of Y is a coarse retract of Y if the inclusion map $i\colon Z \hookrightarrow Y$ is left-invertible, *i.e.* there exists a coarsely Lipschitz map $r\colon Y \longrightarrow Z$ so that $r \circ i \sim \mathrm{Id}_Z$.

Retractions provide an alternative characterisations of coarse expansiveness and coarse left-invertibility.

Proposition 2.19. *Let X, Y be pseudo-metric spaces and $f\colon X \longrightarrow Y$ be a coarsely Lipschitz map. Denote $f_{im}\colon X \longrightarrow f(X)$ the map induced by f. Then the following holds.*

(i) *The map f is coarsely expansive if and only if f_{im} is a metric coarse equivalence.*

(ii) *The map f is coarsely left-invertible if and only if it is coarsely expansive and $f(X)$ is a coarse retract.*

Proof. (i) directly follows from the definitions.

(ii) Suppose that f is coarsely left-invertible and let $g\colon Y \longrightarrow X$ be a coarsely Lipschitz map so that $g \circ f \sim \mathrm{Id}_X$. Denote i the natural inclusion of $f(X)$ into Y. It follows from Remark 2.17 that $\overline{f}$ is a monomorphism, so f is coarsely expansive by Proposition 2.15. Now we have

$$\mathrm{Id}_{f(X)} \circ f_{\mathrm{im}} = f \circ \mathrm{Id}_X \sim f_{\mathrm{im}} \circ (g \circ i \circ f_{\mathrm{im}})$$

and since f_{im} is a metric coarse equivalence by (i) we conclude that $\mathrm{Id}_X \sim (f_{\mathrm{im}} \circ g) \circ i$, and $f_{\mathrm{im}} \circ g$ is a coarse retraction from Y to $f(X)$.

Conversely, suppose f is coarsely expansive and that $f(X)$ is a coarse retract. Let $r\colon Y \longrightarrow f(X)$ be a coarse retraction. By (i), f_{im} is a metric coarse equivalence, so let $j\colon f(X) \longrightarrow X$ be a coarsely Lipschitz map so that $\overline{j}$ and $\overline{f}$ are inverses of each other. Then one has

$$\begin{aligned}(j \circ r) \circ f &= j \circ (r \circ i) \circ f_{\mathrm{im}} \\ &\sim j \circ \mathrm{Id}_{f(X)} \circ f_{\mathrm{im}} \\ &= j \circ f_{\mathrm{im}} \\ &\sim \mathrm{Id}_X\end{aligned}$$

whence $j \circ r$ is a coarse left inverse for f. □

Definition 2.20. Let X, Y be pseudo-metric spaces. We say that Y is coarsely retractable on X if there is a coarsely right-invertible coarsely Lipschitz map from Y to X, or equivalently if there is a coarsely left-invertible coarsely Lipschitz map from X to Y.

We conclude this part by the analog of Proposition 2.15 in the large-scale category.

Proposition 2.21. *Let X, Y be pseudo-metric spaces, $f\colon X \longrightarrow Y$ a large-scale Lipschitz map, and $\overline{f}$ the corresponding morphism in the large-scale category. The following equivalences hold.*

(i) *If $X \neq \emptyset$, $\overline{f}$ is an epimorphism if and only if f is essentially surjective.*

(ii) *The morphism $\overline{f}$ is a monomorphism if and only if f is large-scale expansive.*

(iii) *The morphism $\overline{f}$ is an isomorphism if and only if f is a quasi-isometry. Moreover, if $X \neq \emptyset$, this holds if and only if $\overline{f}$ is an epimorphism and a monomorphism.*

2.2 Coarse and large-scale properties

Let (X, d_X) is a pseudo-metric space and $c > 0$. If $x, x' \in X$ and $n \geq 0$, a *$c-$path of n steps from x to x' in X* is a sequence

$$x = x_0, x_1, \ldots, x_{n-1}, x_n = x'$$

of points in X so that $d_X(x_{i-1}, x_i) \leq c$ for all $i = 1, \ldots, n$.

Definition 2.22. Let (X, d_X) be a pseudo-metric space and $c > 0$. We say that X is

(i) $c-$coarsely connected if for any pair of points $x, x' \in X$, there is a $c-$path from x to x'.

(ii) $c-$coarsely geodesic if there exists an upper control Φ so that, for any pair of points $x, x' \in X$, there is a $c-$path of at most $\Phi(d_X(x, x'))$ steps from x to x'.

(iii) $c-$large-scale geodesic if there exist $a > 0$, $b \geq 0$ so that for any pair of points $x, x' \in X$, there is a $c-$path of at most $a d_X(x, x') + b$ steps from x to x'.

(iv) $c-$geodesic if for any pair of points $x, x' \in X$, there is a $c-$path $x = x_0, x_1, \ldots, x_n = x'$ so that
$$d_X(x, x') = \sum_{i=1}^{n} d_X(x_{i-1}, x_i).$$

(v) geodesic if for any pair of points $x, x' \in X$ with $d_X(x, x') > 0$, there exists an isometric map $\sigma\colon [0, d_X(x, x')] \longrightarrow X$ so that $\sigma(0) = x$ and $\sigma(d_X(x, x')) = x'$.

We say that X is coarsely connected (resp. coarsely geodesic, large-scale geodesic) if it is $c-$coarsely connected (resp. $c-$coarsely geodesic, $c-$large-scale geodesic) for some $c > 0$.

Remark 2.23. (i) Clearly, we have

$$\begin{aligned} X \text{ geodesic} &\Longrightarrow X\ c\text{-geodesic} \\ &\Longrightarrow X \text{ large-scale geodesic} \\ &\Longrightarrow X \text{ coarsely geodesic} \\ &\Longrightarrow X \text{ coarsely connected.} \end{aligned}$$

(ii) If X is $c-$coarsely connected (resp. $c-$coarsely geodesic, $c-$large-scale geodesic, $c-$geodesic) for some $c > 0$, then X is $C-$coarsely connected (resp. $C-$coarsely geodesic, $C-$large-scale geodesic, $C-$geodesic) for any $C \geq c$.

Proposition 2.24. *Coarse connectedness, coarse geodesicity (resp. large-scale geodesicity) are invariant under metric coarse equivalences (resp. quasi-isometries).*

Proof. We show the proof for large-scale geodesicity, and the others are completely similar. Suppose that $f\colon X \longrightarrow Y$ is a $(C,K)-$quasi-isometry, with $C \geq 1$ and $K \geq 0$, and let $c > 0$ be so that X is $c-$large scale geodesic and any point of Y is at distance at most c from $f(X)$. Let $y, y' \in Y$ and let $x, x' \in X$ be so that

$$d_X(y, f(x)),\ d_X(y', f(x')) \leq c.$$

As X is $c-$large scale geodesic, there exist $a > 0$, $b \geq 0$ and a $c-$path

$$x = x_0, x_1, \ldots, x_n = x'$$

so that $n \leq a d_X(x, x') + b$. Set

$$y_0 := y, y_1 := f(x_1), y_2 := f(x_2), \ldots, y_{n-1} := f(x_{n-1}), y_n := y'.$$

Then one has

$$\begin{aligned} d_Y(y_{i-1}, y_i) &= d_Y(f(x_{i-1}), f(x_i)) \\ &\leq C d_X(x_{i-1}, x_i) + K \\ &\leq C \cdot c + K \end{aligned}$$

for all $i = 2, \ldots, n-1$, and also

$$\begin{aligned} d_Y(y_0, y_1) &= d_Y(y, f(x_1)) \\ &\leq d_Y(y, f(x)) + d_Y(f(x_0), f(x_1)) \\ &\leq c + (C \cdot c + K) \\ &= (C+1)c + K \end{aligned}$$

and

$$\begin{aligned} d_Y(y_{n-1}, y_n) &= d_Y(f(x_{n-1}), y') \\ &\leq d_Y(f(x_{n-1}), f(x_n)) + d_Y(f(x'), y') \\ &\leq c + (C \cdot c + K) \end{aligned}$$

$$= (C+1)c + K.$$

Thus $y = y_0, y_1, \dots, y_n = y'$ is a $((C+1)c+K)$–path between y and y' in Y, of at most

$$n \leq a d_X(x,x') + b \leq a(C d_Y(f(x), f(x')) + CK) + b \leq aC(2c + d_Y(y,y')) + aCK + b$$

steps, and the latter is indeed an affine upper bound on the length n of the path in term of the distance $d_Y(y,y')$ between y and y'. As $y, y' \in Y$ were arbitrary, it follows that Y is large-scale geodesic as well. □

In fact, we have the following characterization of those three properties.

Proposition 2.25. *Let (X, d_X) be a pseudo-metric space. The following claims hold.*

(i) *X is coarsely connected if and only if X is coarsely equivalent to a connected metric space.*

(ii) *X is coarsely geodesic if and only if it is coarsely equivalent to a geodesic metric space.*

(iii) *X is large-scale geodesic if and only if it is quasi-isometric to a geodesic metric space.*

The proof of this proposition relies on the next construction and its basic properties.

Definition 2.26. Let $c > 0$, and let (X, d_X) be a c–coarsely connected pseudo-metric space. Let $(X_{\text{Haus}}, d_{\text{Haus}})$ be the largest Hausdorff quotient of X, *i.e.* the quotient of X by the equivalence relation R defined as $xRy \iff d_X(x,y) = 0$. Let X_c denote the connected graph with vertex set X_{Haus}, in which edges connect pairs $(x,y) \in X_{\text{Haus}} \times X_{\text{Haus}}$ with $0 < d_{\text{Haus}}(x,y) \leq c$. Let d_c be the combinatorial metric on X_c, with edges of length c.

Note that, by construction, (X_c, d_c) is geodesic (hence connected).

Consider now the natural map $\varphi\colon (X, d_X) \longrightarrow (X_c, d_c)$, $x \longmapsto [x]$.

Lemma 2.27. *Let $c > 0$, (X, d_X) a c–coarsely connected pseudo-metric space. The natural map $\varphi\colon (X, d_X) \longrightarrow (X_c, d_c)$ has the following properties.*

(i) *For all $x, y \in X$, $d_X(x,y) \leq d_c([x],[y])$. In particular, φ is large-scale expansive.*

(ii) *The map φ is essentially surjective, and $\sup_{w \in X_c} d_c(w, \varphi(X)) \leq \frac{c}{2}$.*

(iii) *If (X, d_X) is coarsely geodesic, then φ is coarsely Lipschitz, and thus is a metric coarse equivalence.*

(iv) *If (X, d_X) is large-scale geodesic, then φ is large-scale expansive, and thus is a quasi-isometry.*

Proof. All claims are straightforward, so we only show the proof of (iii). Assume X is c−coarsely geodesic, and fix two points $x, y \in X$. Then one finds a c−path

$$x = x_0, x_1, \dots, x_{n-1}, x_n = y$$

and an upper control $\Phi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ so that $n \leq \Phi(d_X(x, y))$. Hence we get

$$\begin{aligned} d_c(\varphi(x), \varphi(y)) &= d_c([x_0], [x_n]) \\ &\leq d_c([x_0], [x_1]) + \cdots + d_c([x_{n-1}], [x_n]) \\ &= nc \\ &\leq c\Phi(d_X(x, y)) \\ &= \widetilde{\Phi}(d_X(x, y)) \end{aligned}$$

where $\widetilde{\Phi} := c\Phi$. Thus φ is coarsely Lipschitz as claimed. □

Proof of Proposition 2.25. For (i) we refer to [5, proposition 3.B.7], that uses a slight modification of the space (X_c, d_c).

(ii) If X is coarsely geodesic, then it is coarsely equivalent to a geodesic metric space by Lemma 2.27(iii). Conversely, if it is coarsely equivalent to a geodesic metric space, it is coarsely geodesic as a direct consequence of Remark 2.23 and Proposition 2.24.

(iii) is proved in the same way as (ii), using this time Lemma 2.27(iv). □

Large-scale geodesicity can also be used to boost coarse properties for maps to large-scale properties.

Proposition 2.28. *Let X, Y be pseudo-metric spaces and $f\colon X \longrightarrow Y$ a map.*

(i) *If X is large-scale geodesic and f is coarsely Lipschitz, then f is large-scale Lipschitz.*

(ii) *If X, Y are large-scale geodesic and f is a metric coarse equivalence, then f is a quasi-isometry.*

Proof. (i) Assume that X is c−large-scale geodesic, and let $a > 0$, $b \geq 0$ be so that any pair of points $x, x' \in X$ can be joined by a c−path of at most $a d_X(x, x') + b$ steps. As f is coarsely Lipschitz, we find $C \geq 0$ so that

$$d_X(x, x') \leq c \Longrightarrow d_Y(f(x), f(x')) \leq C. \tag{3}$$

Let $x, x' \in X$ and choose a c−path $x = x_0, x_1, \dots, x_n = x'$ from x to x' of at most $n \leq a d_X(x, x') + b$ steps. Then

$$\begin{aligned} d_Y(f(x), f(x')) &\leq \sum_{i=1}^{n} d_Y(f(x_{i-1}), f(x_i)) \\ &\leq Cn \\ &\leq C(a d_X(x, x') + b) \end{aligned}$$

$$= (aC)d_X(x, x') + bC$$

where the first inequality follows from the triangle inequality and the second one follows from (3). Hence f is large-scale Lipschitz.

(ii) follows directly from (i) applied to f and to $g\colon Y \longrightarrow X$ a metric coarse equivalence so that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$. □

2.3 Groups as pseudo-metric spaces

In this part, we use results of Chapter 1 to explain how topological groups can be seen as objects in the metric coarse category. Additionally, this provides numerous examples of metric coarse equivalences and quasi-isometries.

Definition 2.29. Let G be a topological group. A pseudo-metric d on G is adapted if it is left-invariant, proper, and locally bounded.

Since a topological group G is a homogeneous space (Remark 1.15), a pseudo-metric d on G is adapted if it is left-invariant, balls centered at the identity $e \in G$ are relatively compact, and are neighborhoods of $e \in G$ for large enough radius.

We start with a metric characterisation of σ−compactness.

Theorem 2.30. *Let G be a locally compact group. The following claims are equivalent.*

(i) *The group G is σ−compact.*

(ii) *There exists an adapted continuous pseudo-metric on G.*

(iii) *There exists an adapted pseudo-metric on G.*

(iv) *There exists an adapted metric on G.*

Proof. (i) $\Longrightarrow$ (ii) : Assume that G is σ−compact and locally compact. By Theorem 1.34, there is a compact normal subgroup K so that G/K is metrisable. Equivalently, it is first-countable (Theorem 1.32), and as it is also σ−compact, Theorem 1.33 ensures there exists on G/K a left-invariant proper compatible metric $d_{G/K}$. Now the map

$$d\colon G \times G \longrightarrow [0, +\infty), (g, h) \longmapsto d_{G/K}(gK, hK)$$

is an adapted continuous pseudo-metric on G.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (iv) : If d is an adapted pseudo-metric on G, then the map $d'\colon G \times G \longrightarrow [0, +\infty)$ defined by $d'(g, h) = 1 + d(g, h)$ if $g \neq h$ and $d'(g, g) = 0$ is an adapted metric on G.

(iv) $\Longrightarrow$ (i) : Let d be an adapted metric on G. Then

$$G = \bigcup_{n \in \mathbb{N}} \overline{B_d(e, n)}$$

and subsets appearing in this union are compact since d is proper. Thus G is σ−compact, as announced. □

Theorem 2.30 gives other examples of coarsely Lipschitz maps, namely any continuous homomorphism between σ−compact locally compact groups.

Proposition 2.31. *Let G_1, G_2 be two σ−compact locally compact groups, equipped with adapted pseudo-metrics d_1 and d_2 respectively.*

If $f : (G_1, d_1) \longrightarrow (G_2, d_2)$ is a continuous homomorphism, then f is coarsely Lipschitz, and it is coarsely expansive if and only if it is proper.

Proof. Let $R_1 > 0$. By Proposition 2.2, we must find $R_2 > 0$ so that if $g_1, h_1 \in G_1$ have $d_1(g_1, h_1) \le R_1$, then $d_2(f(g_1), f(g_2)) \le R_2$.

The ball $B_1 := \{g \in G_1 : d_1(e_{G_1}, g) \le R_1\}$ is relatively compact since d_1 is proper. As f is continuous, $f(B_1)$ is relatively compact, and thus bounded in (G_2, d_2) since d_2 is locally bounded (see the remark right after Definition 1.3, that applies since G_2 is locally compact). Hence there is $R_2 > 0$ so that

$$f(B_1) \subset B_2 := \{g \in G_2 : d_2(e_{G_2}, g) \le R_2\}.$$

Now, if $g_1, h_1 \in G_1$ have $d_1(g_1, h_1) \le R_1$, then $d_1(e_{G_1}, g_1^{-1}h_1) \le R_1$ by left-invariance, *i.e.* $g_1^{-1}h_1 \in B_1$, whence $f(g_1^{-1}h_1) \in B_2$ by the above inclusion. This means that

$$d_2(e_{G_2}, f(g_1^{-1}h_1)) \le R_2$$

or equivalently $d_2(f(g_1), f(h_1)) \le R_2$ since f is a homomorphism and d_2 is left-invariant. It follows that f is coarsely Lipschitz.

Now, suppose f is coarsely expansive, and let Φ_- be a lower control for f. Consider a compact subset $L \subset G_2$. As d_2 is locally bounded, there is $R_2 > 0$ so that $L \subset B_2$, where B_2 is as above. Let $R_1 := \inf\{R \ge 0 : \Phi_-(R) \ge R_2\}$. Since $\Phi_-(d_1(e_{G_1}, g)) \le d_2(e_{G_2}, f(g))$ for any $g \in G_1$, it follows that

$$f^{-1}(L) \subset f^{-1}(B_2) \subset B_1$$

and since d_1 is proper, B_1 is relatively compact. Thus $f^{-1}(L)$ is contained in a compact set, and since L is closed (it is compact in G_2 which is Hausdorff) and f is continuous, $f^{-1}(L)$ is closed in G_1. We conclude that $f^{-1}(L)$ is compact, and thus f is proper.

Conversely, assume f is proper, and let $R_2 \ge 0$. Then B_2 is compact, and f is proper, so $f^{-1}(B_2)$ is relatively compact in G_1. The pseudo-metric d_1 being locally bounded, we find $R_1 \ge 0$ so that

$$f^{-1}(B_2) \subset B_1 = \{g \in G_1 : d_1(e_{G_1}, g) \le R_1\}.$$

If $g_1, h_1 \in G_1$ are so that $d_1(g_1, h_1) > R_1$, then $g_1^{-1}h_1 \notin B_1$, whence $g_1^{-1}h_1 \notin f^{-1}(B_2)$, *i.e.* $f(g_1)^{-1}f(h_1) \notin B_2$. Hence

$$d_2(f(g_1)^{-1}f(h_1), e_{G_2}) > R_2$$

which amounts to say that $d_2(f(g_1), f(h_1)) > R_2$ by left-invariance of d_2. We conclude from Proposition 2.3 that f is coarsely expansive. □

In particular, we deduce that for a σ−compact locally compact group, the adapted pseudo-metric provided by Theorem 2.30 is unique up to metric coarse equivalence.

Corollary 2.32. *Let G be a σ−compact locally compact group, H a closed subgroup, d_G, d'_G two adapted pseudo-metrics on G, and d_H an adapted pseudo-metric on H. The following hold.*

(i) *The inclusion map $(H, d_H) \hookrightarrow (G, d_G)$ is a coarse embedding.*

(ii) *The identity map $Id_G\colon (G, d_G) \longrightarrow (G, d'_G)$ is a metric coarse equivalence.*

Proof. (i) Note first that H being a closed subgroup of G, it is itself a σ−compact locally compact group, so Theorem 2.30 indeed ensures that d_H exists. Now the natural inclusion $(H, d_H) \hookrightarrow (G, d_G)$ is a continuous homomorphism, hence it is coarsely Lipschitz by Proposition 2.31. It is also a proper map, hence it is also coarsely expansive by the same result. Thus it is a coarse embedding.

(ii) The map $\mathrm{Id}_G\colon (G, d_G) \longrightarrow (G, d'_G)$ is of course essentially surjective, and coarsely Lipschitz, coarsely expansive still by Proposition 2.31. Hence it is a metric coarse equivalence. □

In particular, any σ−compact locally compact group carries an adapted pseudo-metric that makes it an object in the metric coarse category, well-defined up to metric coarse equivalence.

We now turn to a metric characterisation of compact generation.

Definition 2.33. Let G be a topological group. A pseudo-metric d on G is geodesically adapted if it is adapted and (G, d) is large-scale geodesic.

Definition 2.34. Let G be a group and $S \subset G$ a generating set. The word metric defined by S on G is the metric d_S given by

$$d_S(g, h) := \min\{n \geq 0 : \exists s_1, \dots, s_n \in S \cup S^{-1},\ g^{-1}h = s_1 \dots s_n\}$$

for any $g, h \in G$. The corresponding word length is the map $\ell_S\colon G \longrightarrow \mathbb{N}$ defined by

$$\ell_S(g) := d_S(e_G, g)$$

for any $g \in G$.

Given a group G and a generating set $S \subset G$, it is easy to check that the metric space (G, d_S) is 1−geodesic, in particular large-scale geodesic. Moreover, d_S is left-invariant.

The next result shows that if S is compact, then d_S is geodesically adapted.

Proposition 2.35. *Let G be a topological group. The following statements hold.*

(i) *If G is locally compact and has a compact generating set S, then d_S is geodesically adapted.*

(ii) *If G carries an adapted pseudo-metric d so that (G, d) is coarsely connected (this is the case if d is geodesically adapted for instance), then G is locally compact and compactly generated.*

Proof. (i) We already know that d_S is left-invariant and that (G, d_S) is large-scale geodesic. It is moreover proper, since balls of finite radius around e_G are compact as they are finite unions of compact sets. It remains to check local boundedness, *i.e.* that any point of G has a neighborhood of finite diameter. By homogeneity (Remark 1.15), it is enough to check this condition for $e_G \in G$. By Proposition 1.37, there is $n \geq 0$ so that $\overline{S}^n$ is a neighborhood of e_G. Its diameter with respect to d_S is bounded by $2n$, which concludes the proof of (i).

(ii) Suppose that d is an adapted pseudo-metric on G, and let $c > 0$ be so that (G, d) is c−coarsely connected. Then the ball $B_d(e_G, c)$ is relatively compact (as d is proper), and thus its closure is compact, and is a generating set for G as (G, d) is c−coarsely connected. Thus G is compactly generated, and its local compactness follows from local boundedness of d. □

Next, we want to ensure that the choice of a compact generating set for a compactly generated group does not affect its large-scale geometry.

Lemma 2.36. *Let G be a compactly generated locally compact group, and let $S, T \subset G$ be two compact generating sets.*

Then the identity map $(G, d_S) \longrightarrow (G, d_T)$ is a bilipschitz equivalence.

Proof. As G is locally compact, we may apply Proposition 1.37(iii) and choose $k, \ell \in \mathbb{N}$ so that $T \subset \overline{S}^k$, $S \subset \overline{T}^\ell$. Thus

$$c := \sup_{t \in T} d_S(e_G, t), \ c' := \sup_{s \in S} d_T(e_G, s)$$

are two finite constants. Let $g, h \in G$, let $n := d_S(g, h)$, and let $s_1, \ldots, s_n \in S \cup S^{-1}$ so that $g^{-1}h = s_1 \ldots s_n$. Then it follows that

$$\begin{aligned}
d_T(g, h) &= d_T(e_G, g^{-1}h) \\
&= d_T(e_G, s_1 \ldots s_n) \\
&\leq d_T(e_G, s_1) + d_T(s_1, s_1 \ldots s_n) \\
&= d_T(e_G, s_1) + d_T(e_G, s_2 \ldots s_n) \\
&\leq d_T(e_G, s_1) + d_T(e_G, s_2) + d_T(s_2, s_2 \ldots s_n) \\
&\leq d_T(e_G, s_1) + d_T(e_G, s_2) + \cdots + d_T(e_G, s_n) \\
&\leq c'n \\
&= c'd_S(g, h)
\end{aligned}$$

using n times the triangle inequality and the left-invariance of d_T. By symmetry, it follows that $\frac{1}{c}d_S(g, h) \leq d_T(g, h)$ for any $g, h \in G$. We conclude that

$$\frac{1}{c}d_S(g, h) \leq d_T(g, h) \leq c'd_S(g, h)$$

for any $g, h \in G$, so that $\mathrm{Id}_G \colon (G, d_S) \longrightarrow (G, d_T)$ is a bilipschitz equivalence. □

Lastly, we derive from our previous results a metric characterisation of compact generation for σ−compact locally compact groups.

Theorem 2.37. *Let G be a σ−compact locally compact group, equipped with d an adapted pseudo-metric. The following claims are equivalent.*

(i) *The group G is compactly generated.*

(ii) *The pseudo-metric space (G, d) is coarsely connected.*

(iii) *The pseudo-metric space (G, d) is coarsely geodesic.*

(iv) *There exists a geodesically adapted pseudo-metric on G.*

(v) *There exists a geodesically adapted metric on G.*

In particular, among σ−compact locally compact groups, compact generation is invariant under metric coarse equivalence.

Proof. (i) $\Longrightarrow$ (iii) : Suppose G is compactly generated, and let $S \subset G$ be a compact generating set. As observed above, (G, d_S) is large-scale geodesic, in particular coarsely geodesic, and the map

$$\mathrm{Id}_G : (G, d) \longrightarrow (G, d_S)$$

is a metric coarse equivalence by Corollary 2.32, which applies since G is σ−compact and locally compact. Thus (G, d) is coarsely geodesic as well by Proposition 2.24, which shows (iii).

(iii) $\Longrightarrow$ (ii) is Remark 2.23(i).

(ii) $\Longrightarrow$ (i) is Proposition 2.35(ii).

(i) $\Longrightarrow$ (iv) is Proposition 2.35(i).

(iv) $\Longrightarrow$ (i) again is Proposition 2.35(ii), noting that a geodesically adapted pseudo-metric d on G makes the pair (G, d) coarsely connected.

Thus, so far, we showed that the first four points of the statement are equivalent. It remains to prove that (i) $\Longrightarrow$ (v). The implication (v) $\Longrightarrow$ (iv) is obvious, so that indeed (v) $\Longrightarrow$ (i), and (i) $\Longrightarrow$ (v) is once again Proposition 2.35(i), noting that the word pseudo-metric d_S coming from a compact generating set S is actually a true metric.

In particular, among σ−compact locally compact groups, compact generation is invariant under metric coarse equivalence as a consequence of Proposition 2.24. □

Here are the analogs of Proposition 2.31 and Corollary 2.32 in the large-scale category.

Proposition 2.38. *Let G_1, G_2 be two compactly generated locally compact groups, equipped with two geodesically adapted pseudo-metrics d_1, d_2 respectively.*

If $f : (G_1, d_1) \longrightarrow (G_2, d_2)$ is a continuous homomorphism, then f is large-scale Lipschitz.

Moreover, any metric coarse equivalence $(G_1, d_1) \longrightarrow (G_2, d_2)$ is a quasi-isometry.

Proof. Let $f: (G_1, d_1) \longrightarrow (G_2, d_2)$ be a continuous homomorphism. Then f is coarsely Lipschitz by Proposition 2.31, and (G_1, d_1), (G_2, d_2) are both large-scale geodesic. Thus Proposition 2.24(i) ensures that f is large-scale Lipschitz.

The second statement follows from Proposition 2.24(ii). □

In the discrete setting, if G, H are finitely generated groups endowed with word metrics coming from finite generating sets, we deduce from Proposition 2.38 that any homomorphism $f: G \longrightarrow H$ is a quasi-isometric embedding, and that any group isomorphism from G to H is a quasi-isometry. In particular, any $f \in \mathrm{Aut}(G)$ is a quasi-isometry.

Corollary 2.39. *Let G be a compactly generated locally compact group, equipped with two geodesically adapted pseudo-metrics d_G, d'_G.*

Then the identity map

$$\mathrm{Id}_G: (G, d_G) \longrightarrow (G, d'_G)$$

is a quasi-isometry.

In particular, any compactly generated locally compact carries a geodesically adapted pseudo-metric that makes it an object in the large-scale category, well-defined up to quasi-isometry.

3. Coarse geometric invariants

The goal of this section is to develop powerful tools to be able to distinguish pseudo-metric spaces up to coarse embedding (resp. quasi-isometry), or on the other hand to establish that two given pseudo-metric spaces are coarsely equivalent (resp. quasi-isometric).

3.1 The Milnor-Schwarz lemma

The first result we present is a sufficient criterion to exhibit a quasi-isometry between a group and a space on which it acts. Let us first introduce relevant terminologies.

Consider a topological group G, a non-empty pseudo-metric space (X, d_X), and an action $\alpha\colon G \times X \longrightarrow X$, $(g, x) \longmapsto gx$. For $x \in X$ and $R \geq 0$, denote by $i_x\colon G \longrightarrow X$, $g \longmapsto gx$ the orbit map and

$$S_{x,R} := \{g \in G : d_X(gx, x) \leq R\} = i_x^{-1}(B_{d_X}(x, R)).$$

Definition 3.1. The action $\alpha\colon G \times X \longrightarrow X$ is

(i) faithful if for any $g \neq e_G \in G$ there is $x \in X$ with $gx \neq x$.

(ii) isometric if $d_X(gx, gx') = d_X(x, x')$ for any $g \in G$, $x, x' \in X$.

(iii) metrically proper if $S_{x,R}$ is relatively compact for all $x \in X$ and $R \geq 0$.

(iv) cobounded if there exists a subset $F \subset X$ of finite diameter so that

$$X = \bigcup_{g \in G} gF.$$

(v) locally bounded if for any $g \in G$ and any bounded subset $B \subset X$, there is a neighborhood V of g in G so that VB is bounded in X.

(vi) geometric if it is isometric, metrically proper, cobounded and locally bounded.

If X is moreover locally compact, the action is *proper* if

$$\{g \in G : gL \cap L \neq \emptyset\}$$

is relatively compact in G for any compact subset $L \subset X$, and *cocompact* if there exists a compact $F \subset X$ with

$$X = \bigcup_{g \in G} gF.$$

Some observations are in order here.

Remark 3.2. (i) Let $\alpha\colon G \times X \longrightarrow X$ be a continuous action of a locally compact group G on a proper metric space (X, d_X). Then this action is metrically proper if and only if it is proper.

Proof. Suppose α is metrically proper, and take $L \subset X$ compact. Let $R > 0$. The collection $\{B_{d_X}(x,R) : x \in L\}$ is an open cover of L, which is compact, so there exist $x_1, \dots, x_n \in L$ so that
$$L \subset B_{d_X}(x_1,R) \cup \cdots \cup B_{d_X}(x_n,R).$$
If now $g \in G$ is so that $gL \cap L \neq \emptyset$, let $z \in gL \cap L$ and write $z = gx = y$ for some $x, y \in L$. By the above inclusion, $x \in B_{d_X}(x_i,R)$ and $y \in B_{d_X}(x_j,R)$ for some $1 \leq i,j \leq n$. Then
$$d_X(gx,x) = d_X(y,x) \leq d_X(y,x_j) + d_X(x_j,x_i) + d_X(x_i,x) \leq 2R + \mathrm{diam}(L)$$
so that $g \in S_{x,2R+\mathrm{diam}(L)}$, which is relatively compact as α is metrically proper. Thus $\{g \in G : gL \cap L \neq \emptyset\}$ is contained in a relatively compact set, hence it is itself relatively compact. We conclude that α is proper.

Conversely, if $x \in X$ and $R \geq 0$, then $B_{d_X}(x,R)$ is relatively compact as X is proper, thus contained in a compact set $L \subset X$, whence
$$S_{x,R} = i_x^{-1}(B_{d_X}(x,R)) \subset i_x^{-1}(L) = \{g \in G : gx \in L\} \subset \{g \in G : gL \cap L \neq \emptyset\}.$$
As the action is proper, the right most set above is relatively compact in G, so $S_{x,R}$ is relatively compact in G as well. □

(ii) Under the same assumptions as in (i), α is cobounded if and only if it is cocompact.

Proof. If α is cobounded, there is a set $F \subset X$ of finite diameter with
$$X = \bigcup_{g \in G} gF.$$
Since X is proper, F is relatively compact, so $\overline{F}$ is compact in X, and since also
$$X = \bigcup_{g \in G} g\overline{F}$$
we deduce that α is cocompact.

Conversely, if $F \subset X$ is a compact set whose translates cover X, then F has finite diameter since X is proper, thus the action is cobounded. □

(iii) If $\alpha \colon G \times X \longrightarrow X$ is a continuous action of a topological group G on a metric space (X, d_X) and that d_X is locally bounded, then the action is locally bounded.

The Milnor-Schwarz lemma takes then the following form.

Theorem 3.3. *Let G be a locally compact group, acting geometrically on (X, d_X) a non-empty pseudo-metric space. Let $x \in X$. Define $d_G \colon G \times G \longrightarrow [0,+\infty)$ by $d_G(g,h) := d_X(gx, g'x)$.*

Then d_G is an adapted pseudo-metric on G, and the orbit map
$$\begin{aligned} i_x \colon (G, d_G) &\longrightarrow (X, d_X) \\ g &\longmapsto gx \end{aligned}$$
is a quasi-isometry. In particular, G is $\sigma-$compact. Moreover, if (X, d_X) is coarsely connected, then G is compactly generated.

Proof. The fact that d_G is a pseudo-metric follows from the fact that d_X is a pseudo-metric. For the left-invariance, let $g', g, h \in G$, and note that

$$d_G(g'g, g'h) = d_X(g'gx, g'hx) = d_X(gx, hx) = d_G(g, h)$$

since the action of G on X is isometric. For the properness of d_G, note that

$$B_{d_G}(e_G, R) = \{g \in G : d_G(e_G, g) \leq R\} = \{g \in G : d_X(x, gx) \leq R\} = S_{x,R}$$

is relatively compact since the action is metrically proper. Thus d_G is proper. Lastly, if $g \in G$, then there is a neighborhood V of g in G so that $V\{x\}$ is bounded in X, as the action is locally bounded. Since for any $g', h' \in G$ we have

$$d_G(g', h') = d_X(g'x, h'x)$$

and since $V\{x\}$ is bounded, it follows that V has finite diameter. Hence d_G is locally bounded, and it is therefore an adapted pseudo-metric on G, for which the orbit map is an isometric map. As furthermore the action is cobounded, i_x is essentially surjective, hence it is a quasi-isometry. In particular, G is σ−compact by Theorem 2.30. If moreover (X, d_X) is coarsely connected, then (G, d_G) is coarsely connected by Proposition 2.24, so that G is compactly generated by Theorem 2.37. □

We can then characterise locally compact compactly generated groups as those groups acting geometrically on geodesic metric spaces.

Corollary 3.4. *Let G be a topological group. The following claims are equivalent.*

(i) *The group G is locally compact and compactly generated.*

(ii) *There exists a geometric action of G on a non-empty coarsely geodesic metric space.*

(iii) *There exists a geometric action of G on a non-empty geodesic metric space.*

(iv) *There exists a geometric faithful action of G on a non-empty geodesic metric space.*

Proof. (i) $\Longrightarrow$ (ii) : Assume first that G is locally compact and compactly generated. Choose d an adapted pseudo-metric on G. Then (G, d) is coarsely geodesic by Theorem 2.37, and it is easy to check that the action of G on (G, d) by left multiplication is geometric.

(ii) $\Longrightarrow$ (iii) : Assume that G is a geometric action on a non-empty coarsely geodesic metric space (X, d), and let $c > 0$ be so that pairs of points in X can be joined by c−paths. Let (X_c, d_c) be the metric graph given in Lemma 2.27. Recall that there is a metric coarse equivalence $(X, d) \longrightarrow (X_c, d_c)$, and note that the action of G on (X, d) has a natural extension to an action of G on (X_c, d_c). This action now satisfies (iii).

(iii) $\Longrightarrow$ (iv) : Assume that G acts geometrically on (X, d_X) a non-empty geodesic metric space. Let B denote the unit ball of $\ell^2(G, \mu)$, where μ is a Haar measure on G. The metric space (B, d_B), where

$$d_B(b, b') := \|b - b'\|, \; b, b' \in B$$

is geodesic, and the natural action of G on $\ell^2(G,\mu)$ induces a faithful continuous isometric action of G on (B, d_B). Set now $Y = X\times B$, equipped with the product metric d_Y defined by

$$d_Y((x,b),(x',b'))^2 = d_X(x,x')^2 + d_B(b,b')^2,\ x,x' \in X,\ b,b' \in B.$$

Now the diagonal action of G on Y satisfies (iv).

(iv) $\Longrightarrow$ (i) : Assume that G acts faithfully and geometrically on a geodesic metric space (Y, d_Y). Fix $y_0 \in Y$ and define a pseudo-metric on G by $d(g,g') := d_Y(gy_0, g'y_0)$, for any $g, g' \in G$. Then d is adapted and (G,d) is coarsely connected, so that G is locally compact and compactly generated by Theorem 2.37. $\square$

We also formulate a version of Theorem 3.3 for discrete groups.

Corollary 3.5. *Let G be a group acting isometrically, properly and cocompactly on a non-empty proper geodesic metric space X.*

Then G is finitely generated and quasi-isometric to X, and for any $x \in X$ the orbit map $G \longrightarrow X$, $g \longmapsto gx$ is a quasi-isometry.

Example 3.6. (i) For any $n \geq 1$, the natural action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ is isometric, proper and cocompact. As $\mathbb{R}^n$ is geodesic and proper, it follows that $\mathbb{Z}^n \sim_{Q.I.} \mathbb{R}^n$, as in Example 2.9(i).

(ii) If G is finitely generated and S is a finite symmetric generating for G, then the natural action of G on its Cayley graph $\mathrm{Cay}(G,S)$ is isometric, proper, and cocompact since it is transitive on the vertices and there are $|S|$ equivalence classes of edges. Thus G is quasi-isometric to $\mathrm{Cay}(G,S)$.

(iii) The Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a,b,c,d : a^2 = b^2 = c^2 = d^2 = 1\rangle$ with respect to $S = \{a,b,c,d\}$ is a $4-$regular tree, and is therefore quasi-isometric to any Cayley graph of F_2. Thus $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ is quasi-isometric to F_2.

We now deduce corollaries of interest to get more examples of pairs of quasi-isometric groups.

The first one completes a discussion in subsection 1.5.

Corollary 3.7. *Let G be a finitely generated group and $H \leqslant G$ a finite index subgroup.*

Then H is finitely generated and quasi-isometric to G.

Proof. Consider S a finite generating set for G and the metric space (G, d_S). Let H acts on (G, d_S) by left-multiplication. This action is isometric, proper, and cocompact since a finite set of representatives of left $H-$cosets is a compact subset of (G, d_S) whose translates by H cover G. Moreover, (G, d_S) is geodesic and proper (balls of finite radius centered at $e_G \in G$ are finite), whence H is finitely generated and quasi-isometric to G by Corollary 3.5. Moreover, a quasi-isometry is given by an arbitrary orbit map, for an arbitrary choice of base point in G. The choice $e_G \in G$ shows that the natural inclusion $H \hookrightarrow G$ is a quasi-isometry. $\square$

Example 3.8. (i) The dihedral group $D_\infty = \langle a,t : a^2 = 1, ata^{-1} = t^{-1}\rangle \cong \mathbb{Z}\rtimes\mathbb{Z}/2\mathbb{Z}$ contains $\mathbb{Z}$ as a finite index subgroup, and thus is quasi-isometric to $\mathbb{Z}$.

(ii) The group $\mathrm{SL}_2(\mathbb{Z})$ contains a finite index subgroup isomorphic to F_2 (see *e.g.* [9, proposition 4.4.2]), so $\mathrm{SL}_2(\mathbb{Z}) \sim_{Q.I.} F_2$.

Corollary 3.9. *Let G be a finitely generated group and $N \triangleleft G$ be a finite normal subgroup.*

Then G is quasi-isometric to G/N.

Proof. The natural action of G on G/N satisfies all assumptions of the Milnor-Schwarz lemma, whence the claim. □

This implies for instance that $\mathrm{SL}_2(\mathbb{Z})$ is quasi-isometric to $\mathrm{PSL}_2(\mathbb{Z})$ since the latter is the quotient of $\mathrm{SL}_2(\mathbb{Z})$ by its center $\{\pm I_2\}$.

For the last application, we need a terminology.

Definition 3.10. Let G and H be two groups. We say that they are

(i) commensurable if they contain finite index subgroups $G' \leqslant G$, $H' \leqslant H$ so that $G' \cong H'$.

(ii) weakly commensurable if they contain finite index subgroups $G' \leqslant G$, $H' \leqslant H$ with finite normal subgroups $N \triangleleft G'$, $M \triangleleft H'$ so that $G'/N \cong H'/M$.

The next statement is then a direct consequence of our previous results.

Corollary 3.11. *Let G be a finitely generated group.*

If H is weakly commensurable to G, then H is finitely generated and quasi-isometric to G.

Proof. Assume H is weakly commensurable to G, and let G', H', N, M be as in Definition 3.10. As G is finitely generated, we deduce from Corollary 3.7 that G' is finitely generated, and thus G'/N is finitely generated (Proposition 1.41). Hence H'/M is finitely generated, so that H' is finitely generated. Thus H is finitely generated, and we have

$$H \sim_{Q.I} H' \sim_{Q.I.} H'/M \cong G'/N \sim_{Q.I.} G' \sim_{Q.I.} G$$

where the first and last quasi-isometries are given by Corollary 3.7, and the second and the third quasi-isometries are given by Corollary 3.9. The proof is complete. □

3.2 Metric lattices in pseudo-metric spaces

Now, we are interested in developing coarse geometric *invariants* in order to distinguish two given pseudo-metric spaces up to coarse embeddings (resp. quasi-isometries).

One such invariant, called *growth*, is introduced in the following section. It is straightforward to define in the discrete setting, but generalizing it to arbitrary pseudo-metric spaces requires an additional tool, called *metric lattices*, which are precisely discrete approximations of a pseudo-metric space. We introduce the relevant terminologies and results in this section.

Definition 3.12. Let $c > 0$. A pseudo-metric space (D, d) is called $c-$uniformly discrete if $d(x, x') \geq c$ for any $x, x' \in D$, $x \neq x'$. The space (D, d) is uniformly discrete if it is $c-$uniformly discrete for some $c > 0$.

Note that a uniformly discrete pseudo-metric space is in fact a metric space.

Definition 3.13. A $c-$metric lattice in X is a subspace L that is $c-$uniformly discrete and cobounded. A subspace $L \subset X$ is a metric lattice if it is a $c-$metric lattice for some $c > 0$.

Remark 3.14. If $L \subset X$ is a metric lattice, then the natural inclusion $L \hookrightarrow X$ is a quasi-isometry. In particular, all metric lattices in X are quasi-isometric.

Such a lattice always exists in a non-empty pseudo-metric space.

Proposition 3.15. *Let X be a non-empty pseudo-metric lattice and $x_0 \in X$. For any $c > 0$, there is a $c-$metric lattice L in X containing x_0 so that*

$$\sup_{x \in X} d(x, L) \leq c.$$

In particular, any pseudo-metric space has a metric lattice. More generally, given M a $c-$uniformly discrete subspace of X, there is a $c-$metric lattice in X containing M.

Proof. Let $c > 0$ and let $M \subset X$ be $c-$uniformly discrete. Apply Zorn's lemma to the collection of subsets $L \subset X$ so that $M \subset L$ and

$$\inf_{\ell \neq \ell' \in L} d(\ell, \ell') \geq c.$$

A maximal element of this collection is precisely a $c-$metric lattice L of X containing M so that

$$\sup_{x \in X} d(x, L) \leq c$$

and the last statement is established. The first one follows with $M = \{x_0\}$. $\square$

The next proposition will also be useful below.

Proposition 3.16. *Let X, Y be pseudo-metric spaces, and let $f : X \longrightarrow Y$ be a map.*

(i) *If f is coarsely expansive, then there exists $c > 0$ so that for any $c-$metric lattice L in X, we have*

$$d_Y(f(\ell), f(\ell')) \geq 1$$

for any $\ell, \ell' \in L$.

(ii) *If f is large-scale Lipschitz, then $f_{|L} : L \longrightarrow X$ is Lipschitz for every metric lattice L in X.*

(iii) *If f is a quasi-isometric embedding, then there exists $k > 0$ so that, for any $k-$metric lattice L in X, $f_{|L} : L \longrightarrow f(L)$ is bilipschitz. In particular, if f is a quasi-isometry, there is $k > 0$ so that $f(L)$ is a metric lattice in Y for any $k-$metric lattice L in X.*

Proof. (i) Assume that f is coarsely expansive, and let Φ_- be a lower control for f. Choose $c > 0$ so that $\Phi_-(c) \geq 1$ (such a c exists as $\lim_{t\to\infty} \Phi_-(t) = \infty$). Then if L is a c−metric lattice in X and $\ell, \ell' \in L$, one gets

$$d_Y(f(\ell), f(\ell')) \geq \Phi_-(d_L(\ell,\ell')) \geq \Phi_-(c) \geq 1$$

as $d_L(\ell,\ell') \geq c$ and Φ_- is non-decreasing.

(ii) directly follows from the fact that a metric lattice is uniformly discrete and Example 2.13(v).

(iii) Suppose $f\colon X \longrightarrow Y$ is a quasi-isometric embedding, so there is $a \geq 1, b \geq 0$ with

$$\frac{1}{a} d_X(x,x') - b \leq d_Y(f(x), f(x')) \leq a d_X(x,x') + b$$

for any $x, x' \in X$. If $b = 0$, then f is bilipschitz on X, so we may suppose that $b > 0$. Let $k := 2ab > 0$, and let L be a k−metric lattice in X. Then for any $\ell, \ell' \in L$ one has

$$\begin{aligned} d_Y(f(\ell), f(\ell')) &\geq \frac{1}{a} d_L(\ell,\ell') - b \\ &\geq \frac{1}{a} d_X(\ell,\ell') - b\frac{d_L(\ell,\ell')}{k} \\ &= \left(\frac{1}{a} - \frac{1}{2a}\right) d_L(\ell,\ell') \\ &= \frac{1}{2a} d_L(\ell,\ell'). \end{aligned}$$

Additionally, since f is large-scale Lipschitz, (ii) ensures that f is Lipschitz on L. Thus we conclude that $f_{|L}\colon L \longrightarrow f(L)$ is bilipschitz. □

3.3 Growth for pseudo-metric spaces

We can define now properly growth for a certain class of pseudo-metric spaces.

Definition 3.17. A pseudo-metric space (X, d_X) is locally finite if all its balls are finite, and uniformly locally finite if

$$\sup_{x\in X} |B_{d_X}(x,r)| < \infty$$

for all $r \geq 0$.

Observe that a uniformly locally finite pseudo-metric space is locally finite.

Example 3.18. (i) A discrete metric space is locally finite if and only if it is proper.

(ii) The real line $X = \mathbb{R}$, with its usual metric, is not uniformly locally finite. The same applies more generally for $X = \mathbb{R}^n$, $n \geq 1$.

(iii) Even more generally, geodesic metric spaces are not locally finite. Hence, for instance, any Banach or Hilbert space is not locally finite.

(iv) If G is a finitely generated group and $S \subset G$ is a finite symmetric generating set for G, then the metric space (G, d_S) is uniformly locally finite.

(v) The vertex set of a connected graph, endowed with the combinatorial metric, is uniformly locally finite if and only if the graph is of bounded degree. If $N \in \mathbb{N}$ is a bound on the degrees of vertices of the graph, then balls of radius n have at most $N(N-1)^{n-1}$ vertices, for all $n \geq 1$. This generalises the previous point, since a finitely generated group is the vertex set of its Cayley graph with respect to a finite symmetric generating set S, for which the combinatorial metric is exactly the word metric d_S.

Definition 3.19. Let (X, d_X) be a locally finite pseudo-metric space, and $x \in X$. The growth function $\beta_X^x \colon \mathbb{R}_+ \longrightarrow \mathbb{N}$ of X around $x \in X$ is defined as

$$\beta_X^x(r) := |B_{d_X}(x, r)|, \ r \geq 0.$$

In order to be able to define an invariant for spaces, we introduce the following equivalence relation.

Definition 3.20. Let $f, f' \colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be non-decreasing. We say that f' dominates f, and we write $f \preccurlyeq f'$, if there exists $\lambda, \mu > 0$, $c \geq 0$ so that

$$f(r) \leq \lambda f'(\mu r + c) + c$$

for any $r \geq 0$. We say that f is equivalent to f', and we write $f \sim f'$, if f dominates f' and if f' dominates f.

It is easy to check that $\sim$ is indeed an equivalence relation, and given a non-decreasing function f on $\mathbb{R}_+$, its *class* refers to its equivalence class modulo the relation $\sim$. If no confusion is possible, we often write f for the class of a function f.

Example 3.21. (i) Let $a, b > 0$ and $c, d > 1$. One has $r^a \preccurlyeq r^b$ if and only if $a \leq b$ (and thus $r^a \sim r^b$ if and only if $a = b$), and $c^r \sim d^r$.

Proof. We start with the first equivalence. If $a \leq b$, then $r^a \leq r^b$ for any $r \geq 0$, so $r^a \preccurlyeq r^b$. Conversely, assume that $r^a \preccurlyeq r^b$, meaning there are $\lambda, \mu > 0$, $c \geq 0$ so that

$$r^a \leq \lambda(\mu r + c)^b + c$$

for any $r \geq 0$. Hence $r^a \leq \lambda r^b(\mu + \frac{c}{r})^b + c$ for any $r \geq 0$, so that

$$r^{a-b} \leq \lambda\big(\mu + \frac{c}{r}\big)^b + \frac{c}{r^b}$$

for any $r \geq 0$. If $a > b$, then letting $r \to \infty$ in the above inequality would provide a contradiction, since the left hand side tends to ∞ while the right hand side tends to a finite value. Thus necessarily $a \leq b$, as claimed.

For the second claim, note that it is enough to prove that if $1 < d \leq c$, then $c^r \preccurlyeq d^r$. Let $\lambda := 1 + \lfloor \frac{\ln(c)}{\ln(d)} \rfloor$, $\mu := \lambda$ and $c := 0$. Then one has $\ln(c) \leq \lambda \ln(d)$, *i.e.* $c \leq d^\lambda$, whence $\frac{c}{d} \leq d^{\lambda-1}$. It follows that

$$\big(\frac{c}{d}\big)^r \leq (d^{\lambda-1})^r \leq \lambda(d^{\lambda-1})^r$$

for any $r \geq 0$, and thus $c^r \leq \lambda(d^{\lambda-1})^r d^r = \lambda(d^r)^\lambda$ for all $r \geq 0$. We conclude that $c^r \preccurlyeq d^r$. $\square$

(ii) For any $a > 0, b > 1$, $r^a \preccurlyeq b^r$ and $r^a \nsim b^r$. The first part is proved exactly in the same spirit as the previous example. On the other hand, if we suppose that $r^a \sim b^r$, then we find $\lambda, \mu > 0$ and $c \geq 0$ so that

$$b^r \leq \lambda(\mu r + c)^a + c$$

for any $r > 0$. Equivalently, $\frac{b^r}{r^a} \leq \lambda(\mu + \frac{c}{r})^a + \frac{c}{r^a}$ for any $r > 0$. Letting $r \longrightarrow \infty$ in this inequality provides a contradicting since the left hand side tends to ∞ while the right hand side tends to a finite value. Hence $r^a \nsim b^r$.

Observe now that, in a uniformly locally finite pseudo-metric space (D, d), if $x, y \in D$ and $r \geq 0$, then

$$B_d(y, r) \subset B_d(x, r + d(x, y)),\ B_d(x, r) \subset B_d(y, r + d(x, y))$$

by the triangle inequality, so that

$$\beta_D^x(r) = |B_d(x, r)| \leq |B_d(y, r + d(x, y))| = \beta_D^y(r + d(x, y))$$

for any $r \geq 0$, and also $\beta_D^y(r) \leq \beta_D^x(r + d(x, y))$ for any $r \geq 0$. Hence $\beta_D^x \sim \beta_D^y$, which motivates the next definition.

Definition 3.22. Let D be a uniformly locally finite pseudo-metric space. Let $x \in D$. The growth type of D is the class of the function β_D^x, and is denoted β_D.

If $\beta_D(r) \sim r^d$ for some $d \in \mathbb{N}$, we say that D has *polynomial growth* of degree d, and we say it has *exponential growth* if $\beta_D(r) \sim e^r$. It has *subexponential growth* if $\beta_D(r) \preccurlyeq e^r$ and $\beta_D(r) \nsim e^r$, and it has *superpolynomial growth* if, for any $d \in \mathbb{N}$, $r^d \preccurlyeq \beta_D(r)$ and $r^d \nsim \beta_D(r)$. Lastly, D has *intermediate growth* if it has superpolynomial growth and subexponential growth.

Since the function β_D^x heavily depends on the metric, we will sometimes write $\beta_{(D,d)}^x$ to insist on the metric we choose on D.

We now give several examples of growth functions, mostly among finitely generated groups.

Example 3.23. (i) Let $G = \mathbb{Z}$ equipped with the metric d_S where $S = \{-1, 1\}$. Then, if $n \geq 1$, $B_{d_S}(0, n) = \{-n, -n+1, \dots, n-1, n\}$, so that $\beta_{(\mathbb{Z},S)}(n) = 2n + 1$ for any $n \geq 1$.

(ii) Let $(\mathbb{Z}, d_S)$ be as in (i), and let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $f(x) = \ln(1 + \ln(1 + x))$. Define on $\mathbb{Z}$ a metric d_f by

$$d_f(x, x') := f(d_S(x, x')),\ x, x' \in \mathbb{Z}.$$

Then $(\mathbb{Z}, d_f)$ is uniformly locally finite, since translations are isometries. Since we have

$$n \in B_{d_f}(0, r) \Longleftrightarrow f(|n|) \leq r \Longleftrightarrow 1 + |n| \leq e^{e^r - 1} \Longleftrightarrow |n| \leq e^{e^r - 1} - 1$$

for any $r \geq 0$, it follows that $\beta^0_{(\mathbb{Z},d_f)}(r) = 2\lfloor e^{e^r - 1} \rfloor - 1$ for any $r \geq 0$.

(iii) Let $G = \mathbb{Z}^2$ equipped with the genereating set $S = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$. For $n \geq 0$, the ball of radius n centered at $(0, 0)$ is

$$\{(i, j) \in \mathbb{Z}^2 : |i| + |j| \leq n\}$$

the diagonal square containing the vertices $(0,r),(0,-r),(r,0),(-r,0)$, $0 \le r \le n$. Thus

$$\beta_{(\mathbb{Z}^2,S)}(n) = 1 + \sum_{r=1}^{n} 4r = 2n^2 + 2n + 1$$

for all $n \ge 0$.

(iv) If $G = \mathbb{Z}^2$ is rather endowed with $S' = S \cup \{(1,1),(-1,-1),(1,-1),(-1,1)\}$, then for any $n \ge 0$, the ball of radius n centered at $(0,0)$ is now

$$\{(i,j) \in \mathbb{Z}^2 : |i| \le n, |j| \le n\} = \{-n,\dots,n\}^2$$

so that $\beta_{(\mathbb{Z}^2,S')}(n) = (2n+1)^2 = 4n^2 + 4n + 1$ for any $n \ge 0$.

(v) Let $G = F_2$ equipped with $S = \{a, b, a^{-1}, b^{-1}\}$. For all $n \ge 1$, the ball of radius n centered at the identity element has cardinality

$$\beta_{(F_2,S)}(n) = 1 + 4\sum_{j=0}^{n-1} 3^j = 2\cdot 3^n - 1.$$

Proposition 3.24. *Let D, E be non-empty uniformly locally finite pseudo-metric spaces, and let $x \in D$ and $y \in E$.*

(i) *If $c > 0$ and L is a $c-$metric lattice containing x, then $\beta_L^x \sim \beta_D^x$ and $\beta_L = \beta_D$.*

(ii) *If there is $c, c'' > 0$ and an injective map $f : D \hookrightarrow E$ so that*

$$c'' d_D(x', x'') \le d_E(f(x'), f(x'')) \le c d_D(x', x'')$$

for any $x', x'' \in D$, then $\beta_D^x \preccurlyeq \beta_E^y$ and $\beta_D \preccurlyeq \beta_E$. If moreover $f(D)$ is cobounded in E, then $\beta_D^x \sim \beta_E^y$ and $\beta_D = \beta_E$.

(iii) *If D, E are quasi-isometric, then $\beta_D^x \sim \beta_E^y$ and $\beta_D = \beta_E$.*

Proof. (i) Let $c, c' > 0$ so that L is a $c-$metric lattice and so that any point of X is at distance at most c' from a point of L. As $L \subset D$, we already have $\beta_L^x \preccurlyeq \beta_D^x$.

Conversely, let $r \ge 0$ and $z \in B_D(x,r)$. Pick some $x' \in L$ so that $d(z,x') \le c'$, and thus one has

$$d(x,x') \le d(x,z) + d(z,x') \le r + c'$$

so that $x' \in B_L(x, r+c')$ and $z \in B_D(x', c')$. Hence

$$B_D(x,r) \subset \bigcup_{x' \in B_L(x,r+c')} B_D(x',c')$$

and taking cardinals, it follows that $\beta_D^x(r) \le \lambda \beta_L^x(r+c')$ with $\lambda := \sup_{x' \in D} |B_D(x',c')| < \infty$. We conclude that $\beta_D^x \preccurlyeq \beta_L^x$, as claimed.

(ii) Let $r \geq 0$ and let $c' := d_E(f(x), y)$. The restriction of f to $B_D(x, r)$ is an injection into $B_E(f(x), cr)$, which is contained in $B_E(y, cr + c')$. Thus

$$\beta_D^x(r) = |B_D(x, r)| \leq |B_E(y, cr + c')| = \beta_E^y(cr + c')$$

which implies $\beta_D^x \preccurlyeq \beta_E^y$. Assume moreover that $f(D)$ is cobounded, and set

$$s := \sup_{y' \in E} d_E(f(D), y') < \infty.$$

For any $y' \in B_E(y, r)$, there is $x' \in D$ with $d_E(f(x'), y') \leq s$, and in this case

$$d_E(f(x), f(x')) \leq d_E(f(x), y) + d_E(y, y') + d_E(y', f(x')) \leq r + c' + s$$

hence

$$\begin{aligned} B_E(y, r) \subset B_E(f(x), r + c') &\subset \bigcup_{x' \in D,\ d_E(f(x), f(x')) \leq r + c' + s} B_E(f(x'), s) \\ &\subset \bigcup_{x' \in D,\ d_E(x, x') \leq \frac{1}{c}(r + c' + s)} B_E(f(x'), s). \end{aligned}$$

Taking cardinals, it follows that $\beta_E^y(r) \leq \mu \beta_D^x(\frac{1}{c}(r + c' + s))$, where

$$\mu := \sup_{y' \in E} |B_E(y', s)| < \infty.$$

We conclude that $\beta_E^y \preccurlyeq \beta_D^x$, and finally that $\beta_D^x \sim \beta_E^y$.
(iii) Suppose that $f : D \longrightarrow E$ is a quasi-isometry. In particular, f is a quasi-isometric embedding, so Proposition 3.16(iii) provides $k > 0$ so that $f_{|L} : L \longrightarrow f(L)$ is bilipschitz for any $k-$metric lattice L in X. Choose such a metric lattice L containing x, using Proposition 3.15. Then, by (i), it follows that $\beta_D^x \sim \beta_L^x$. Now, notice that $f_{|L}$ is also injective, since it is bilipschitz and since L is actually a metric space. Thus, we are in position to apply (ii) and get $\beta_L^x \sim \beta_E^y$. We conclude that $\beta_D^x \sim \beta_E^y$. □

Thus, among uniformly locally finite pseudo-metric spaces, the growth type is a quasi-isometry invariant. This has already important consequences in the discrete setting.

Corollary 3.25. *Let X, Y be two graphs of bounded degree. If X and Y are quasi-isometric, then they have the same growth type.*

Proof. A graph of bounded degree is a uniformly locally finite pseudo-metric space when endowed with the combinatorial metric (cf. Example 3.18(i)). Hence Proposition 3.24(iii) applies. □

Since a finitely generated is quasi-isometric to any of its Cayley graph, we get the following statements.

Corollary 3.26. *Let G be a finitely generated group, and S, S' be finite symmetric generating sets for G.*

(i) *We have* $\beta_{(G,S)} \sim \beta_{(G,S')}$*, so that the growth type of* G *is independent of the choice of generating sets.*

(ii) *The group* G *grows at most exponentially fast.*

Proof. (i) From Lemma 2.36, there is a bilipschitz equivalence $(G, d_S) \longrightarrow (G, d_{S'})$, so that Proposition 3.24(iii) gives the conclusion.

(ii) By Example 3.6(ii), (G, d_S) is quasi-isometric to $\mathrm{Cay}(G, S)$. Since this graph is of bounded degree, it grows at most exponentially fast, so that the same holds for G. □

Putting together Corollary 3.7, Corollary 3.9, Corollary 3.11 and Proposition 3.24, we also deduce the following from our previous results.

Corollary 3.27. (i) *If* G *is a finitely generated group, and* $H \leqslant G$ *has finite index, then* G *and* H *have the same growth type.*

(ii) *If* G *is finitely generated and* H *is weakly commensurable to* G*, then* G *and* H *have the same growth.*

(iii) *If* $N \lhd G$ *is finite, then* G *and* G/N *have the same growth type.*

For instance, $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{PSL}_2(\mathbb{Z})$, $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ all have exponential growth, since all are quasi-isometric to F_2. On the other hand, D_∞ has polynomial growth of degree 1 since it is quasi-isometric to $\mathbb{Z}$.

Let us also mention the following.

Proposition 3.28. *Let* $G = \langle S \rangle$, $H = \langle T \rangle$ *be two finitely generated groups.*

Then $G \times H$ *is finitely generated, and* $\beta_{G\times H} \sim \beta_{(G,S)}\beta_{(H,T)}$.

Proof. The fact that $G \times H$ is finitely generated is a consequence of Proposition 1.41, and the proof of the latter shows that

$$U := \{(s, e_H) : s \in S\} \cup \{(e_G, t) : t \in T\}$$

is a finite generating set for $G \times H$. As the growth type of $G \times H$ is independent of the choice of the generating set, we now show that $\beta_{(G\times H,U)} \sim \beta_{(G,S)}\beta_{(H,T)}$.

First of all, if $g \in B_{\mathrm{d}_S}(e_G, n)$ and if $h \in B_{\mathrm{d}_T}(e_H, n)$, then $(g, h) \in B_{\mathrm{d}_U}(e_{G\times H}, 2n)$, and thus

$$\begin{aligned}
\beta_{(G,S)}(n)\beta_{(H,T)}(n) &= |B_{\mathrm{d}_S}(e_G, n)||B_{\mathrm{d}_T}(e_H, n)| \\
&\le |B_{\mathrm{d}_U}(e_{G\times H}, 2n)| \\
&= \beta_{(G\times H,U)}(2n) \\
&\le 2\beta_{(G\times H,U)}(2n)
\end{aligned}$$

for any $n \ge 1$. Hence $\beta_{(G,S)}\beta_{(H,T)} \preccurlyeq \beta_{(G\times H,U)}$. Conversely, if $n \ge 1$ and $(g, h) \in B_{\mathrm{d}_U}(e_{G\times H}, n)$, there is $p, r \in \mathbb{N}$ with $p+r \le n$ and group elements $x_1, \ldots, x_p \in S \cup S^{-1}$, $y_1, \ldots, y_r \in T \cup T^{-1}$ so that

$$(g, h) = (x_1, e_H)\ldots(x_p, e_H)(e_G, y_1)\ldots(e_G, y_r) = (x_1 \ldots x_p, y_1 \ldots y_r).$$

Hence $B_{\mathrm{d}_U}(e_{G\times H}, n) \subset B_{\mathrm{d}_S}(e_G, n) \times B_{\mathrm{d}_T}(e_H, n)$, and it follows that

$$\beta_{(G\times H,U)}(n) \leq \beta_{(G,S)}(n)\beta_{(H,T)}(n)$$

for any $n \geq 1$. We conclude that $\beta_{(G\times H,U)} \preccurlyeq \beta_{(G,S)}\beta_{(H,T)}$, and finally that $\beta_{(G\times H,U)} \sim \beta_{(G,S)}\beta_{(H,T)}$. □

Therefore, as $\beta_{\mathbb{Z}} \sim n$ (cf. Example 3.23(i)), it follows that $\mathbb{Z}^d$ has polynomial growth of degree d, for any $d \geq 1$. This implies the following.

Corollary 3.29. *Any finitely generated abelian group has polynomial growth.*

Proof. Such a group G splits as a product $\mathbb{Z}^d \times F$, where $d \in \mathbb{N}$ and F is a finite group (see *e.g.* [3, corollary 1.30]). As the growth function of F is constant for n large enough, we conclude that $\beta_G \sim \beta_{\mathbb{Z}^d} \sim n^d$, as claimed. □

These results already allow us to distinguish euclidean spaces up to quasi-isometry.

Corollary 3.30. *For any $d \neq d' \in \mathbb{N}$, $\mathbb{Z}^d$ is not quasi-isometric to $\mathbb{Z}^{d'}$. As a consequence, $\mathbb{R}^d$ is not quasi-isometric to $\mathbb{R}^{d'}$.*

Proof. (i) If $d \neq d'$ and $\mathbb{Z}^d \sim_{Q.I.} \mathbb{Z}^{d'}$, then $n^d \sim n^{d'}$, so $d = d'$ by Example 3.21(i). This contradiction shows that $\mathbb{Z}^d \nsim_{Q.I.} \mathbb{Z}^{d'}$. In particular, as $\mathbb{R}^d$ (resp. $\mathbb{R}^{d'}$) is quasi-isometric to $\mathbb{Z}^d$ (resp. $\mathbb{Z}^{d'}$), we deduce also $\mathbb{R}^d \nsim_{Q.I.} \mathbb{R}^{d'}$. □

So far, we defined growth for uniformly locally finite pseudo-metric spaces, and in this class, it is a quasi-isometry invariant. It therefore makes sense to define growth for pseudo-metric spaces quasi-isometric to uniformly locally finite pseudo-metric spaces, without requiring them to be uniformly locally finite from the beginning. This property is related to *coarse properness* and *uniform coarse properness*, that we define now.

Definition 3.31. A pseudo-metric space (X, d_X) is coarsely proper if there exists $R_0 \geq 0$ so that any bounded subset of X can be covered by finitely many balls of radius R_0.

Proposition 3.32. *Let (X, d_X) be a pseudo-metric space. The following claims are equivalent.*

(i) *The space (X, d_X) is coarsely proper.*

(ii) *The space (X, d_X) is coarsely equivalent to a locally finite discrete metric space.*

(iii) *The space (X, d_X) is quasi-isometric to a locally finite discrete metric space.*

(iv) *The space (X, d_X) contains a locally finite metric lattice.*

(v) *For any $c > 0$, the space (X, d_X) contains a locally finite $c-$metric lattice.*

(vi) *There exists $c_0 > 0$ so that, for any $c \geq c_0$, every $c-$metric lattice in X is locally finite.*

If moreover X is large-scale geodesic, X satisfies properties (i)-(vi) if and only if (vii) X is quasi-isometric to a locally finite connected graph.

Proof. Note to start that (v) $\Longrightarrow$ (iv) is obvious, that (iv) $\Longrightarrow$ (iii) follows from Remark 3.14, and that (iii) $\Longrightarrow$ (ii) is also clear. We now prove that (i) $\Longleftrightarrow$ (iii).

(i) $\Longrightarrow$ (iii) : Suppose that X is coarsely proper, and let $R_0 \geq 0$ be given by Definition 3.31. Let L be a $3R_0$–metric lattice in X, which exists by Proposition 3.15. By Remark 3.14, X is quasi-isometric to L, so it is enough to show that L is locally finite. Let $F \subset L$ be a bounded subset. Then F is included in the union of finitely many balls $B_1, \ldots, B_n$ of radius R_0, and since the distance between two points of L is at least $3R_0$, the ball B_i can contain at most one point of F, for any $1 \leq i \leq n$. Hence F is finite, and thus balls of L are finite.

(iii) $\Longrightarrow$ (i) : Let $f: (X, d_X) \longrightarrow (Y, d_Y)$ be a quasi-isometry, with Y a locally finite discrete metric space. Denote $a > 0$, $b \geq 0$ the parameters of f. Observe first that the pre-image under f of any singleton in Y has finite diameter: if $y \in Y$ and $x, x' \in f^{-1}(\{y\})$, then

$$d_X(x, x') \leq a(d_Y(f(x), f(x')) + b) = a(d_Y(y, y) + b) = ab.$$

Let then $R_0 := ab \geq 0$, and let $B \subset X$ be bounded. Then $f(B) \subset Y$ is bounded, as

$$d_Y(f(x), f(x')) \leq a d_X(x, x') + b \leq a\,\mathrm{diam}(B) + b$$

for any $x, x' \in B$. As Y is locally finite, we deduce that $f(B)$ is finite, and we enumerate $f(B) = \{f(x_1), \ldots, f(x_n)\}$ for $x_1, \ldots, x_n \in B$. It follows that

$$B \subset f^{-1}(f(B)) = \bigcup_{i=1}^{n} f^{-1}(\{f(x_n)\}) \subset \bigcup_{i=1}^{n} B_{d_X}(x_i, R_0)$$

where the last inclusion is justified by our previous observation. Hence (X, d_X) is coarsely proper.

(ii) $\Longrightarrow$ (iv) : Assume there exists a locally finite discrete metric space (D, d) and a metric coarse equivalence $f: D \longrightarrow X$. By Proposition 3.16(i), we find a metric lattice $L \subset D$ so that

$$d_X(f(\ell), f(\ell')) \geq 1$$

for any $\ell \neq \ell' \in L$. We denote d_L the restriction of d to L. Then $f(L) \subset X$ is a metric lattice in X, so it remains to check it is locally finite. Let $B \subset f(L)$ be bounded. Then $f^{-1}(B)$ is bounded as f is coarsely expansive. Since (L, d_L) is proper, $f^{-1}(B) \cap L$ is finite. Since $f_{|L}: L \longrightarrow X$ is injective, it follows that B is finite. Thus $f(L)$ is a locally finite metric lattice in X, which shows (iv).

(iv) $\Longrightarrow$ (v) : Let $c > 0$. From Proposition 3.15, every metric lattice contains a c–metric lattice, whence the claim.

So far, we have proved the equivalences of properties (i)-(v). Note that (vi) $\Longrightarrow$ (iv) follows once again from Proposition 3.15.

(iv) $\Longrightarrow$ (vi) : Let $D \subset X$ be a locally finite metric lattice. It is cobounded, so let $R > 0$ be so that any point of X is at distance at most R from a point of D. Let $c_0 := 2R + 1$, and fix E a

$c-$metric lattice for $c \geq c_0$. For any $e \in E$, choose an element $d_e \in D$ so that $d_X(e, d_e) \leq R$, and define $f: E \longrightarrow D$ setting $f(e) := d_e$. This map is injective, since if $e \neq e' \in E$ are so that $f(e) = f(e')$, then

$$2R + 1 = c_0 \leq c \leq d_X(e, e') \leq d_X(e, d_e) + d_X(d_e, e') \leq 2R$$

a contradiction. Moreover, if $r \geq 0$ and $e \in E$, the image of $B_{d_E}(e, r)$ under f is contained in $B_{d_D}(f(e), 2R)$. The latter being finite by hypothesis on D, we conclude that $B_{d_E}(e, r)$ is finite as well. Thus E is locally finite.

To conclude, we assume moreover that X is large-scale geodesic, and we show that (iii) $\Longleftrightarrow$ (vii). The implication (vii) $\Longrightarrow$ (iii) is immediate, so we turn to the converse.

(iii) $\Longrightarrow$ (vii) : Assume that X is $c-$large-scale geodesic, and by (iii) without restriction we assume that X is a locally finite discrete metric space. Consider the graph X' for which vertices are elements of X and two such vertices are linked by an edge if they are at distance less than c in X. Endow X' with the natural combinatorial metric d'. As X is $c-$large-scale geodesic, X' is connected, and the map

$$\mathrm{Id}_X : (X, d_X) \longrightarrow (X', d')$$

is a quasi-isometry. As (X, d_X) is locally finite, so is (X', d'), and (vii) is proved. □

In particular, we deduce that coarse properness is a coarse geometric invariant.

Corollary 3.33. *For pseudo-metric spaces, coarse properness is invariant under metric coarse equivalence.*

Proof. This follows directly from (ii) in the previous proposition. □

We also get examples of coarsely proper metric spaces.

Corollary 3.34. *Proper metric spaces are coarsely proper.*

Proof. Let (X, d_X) be a proper metric space. Choose a metric lattice $L \subset X$ such that $d_X(\ell, \ell') > 2$ for any $\ell \neq \ell' \in L$. By Proposition 3.32(iv), it is enough to prove that L is locally finite to deduce that X is coarsely proper.

Thus let $R > 0$, $\ell_0 \in L$, and consider the ball $B := B_L(\ell_0, R)$ in L. If B is infinite, then $(B_{d_X}(\ell, 1) = \{x \in X : d_X(x, \ell) \leq 1\})_{\ell \in B}$ is an infinite collection of pairwise disjoint non-empty balls of radius 1 in X, all contained in $B_{d_X}(\ell_0, R + 1)$, which is relatively compact as X is proper. This contradiction proves that balls of L are finite, as was to be shown. □

Let us now turn to the corresponding uniform notion.

Definition 3.35. A pseudo-metric space (X, d_X) is uniformly coarsely proper if there exists $R_0 \geq 0$ so that, for any $R \geq 0$, there exists $N \in \mathbb{N}$ so that any ball of radius R can be covered by N balls of radius R_0.

Clearly, any uniformly coarsely proper pseudo-metric space is coarsely proper.

Here is the natural analog of Proposition 3.32 and Corollary 3.33 for uniform coarse properness.

Proposition 3.36. *Let (X, d_X) be a pseudo-metric space. The following claims are equivalent.*

(i) *The space (X, d_X) is uniformly coarsely proper.*

(ii) *The space (X, d_X) is coarsely equivalent to a uniformly locally finite discrete metric space.*

(iii) *The space (X, d_X) is quasi-isometric to a uniformly locally finite discrete metric space.*

(iv) *The space (X, d_X) contains a uniformly locally finite metric lattice.*

(v) *For any $c > 0$, the space (X, d_X) contains a uniformly locally finite $c-$metric lattice.*

(vi) *There exists $c_0 > 0$ so that, for any $c \geq c_0$, every $c-$metric lattice in X is uniformly locally finite.*

If moreover X is large-scale geodesic, X satisfies properties (i)-(vi) if and only if (vii) X is quasi-isometric to a connected graph of bounded degree.

Corollary 3.37. *For pseudo-metric spaces, uniform coarse properness is invariant under metric coarse equivalence.*

Hence we can define the growth type of a uniformly coarsely proper pseudo-metric space as the growth type of one of its uniformly locally finite metric lattices. The next statement shows this definition does not depend on the choice of the lattice, nor of the basepoint.

Proposition 3.38. *Let X be a non-empty uniformly coarsely proper pseudo-metric space. Let L_0, L_1 be metric lattices in X with $x_0 \in L_0$, $x_1 \in L_1$. Then one has*

$$\beta_{L_0}^{x_0} \simeq \beta_{L_1}^{x_1}.$$

Proof. By point (vi) of Proposition 3.36, we may assume that L_0, L_1 are uniformly locally finite. Let $\iota_0 \colon L_0 \hookrightarrow X$, $\iota_1 \colon L_1 \hookrightarrow X$ be the natural inclusions. These are quasi-isometries (Remark 3.14), and letting $p_1 \colon X \longrightarrow L_1$ be a quasi-inverse of ι_1, the composition $p_1 \circ \iota_0$ is a quasi-isometry from L_0 to L_1. We thus may apply Proposition 3.24(iii) to get

$$\beta_{L_0}^{x_0} \simeq \beta_{L_1}^{x_1}$$

as claimed. □

Definition 3.39. Let X be a uniformly coarsely proper pseudo-metric space. The growth type of X, denoted β_X, is the class of the function β_L^x, where L a uniformly locally finite metric lattice in X and $x \in L$.

Example 3.40. For any $d \geq 1$, $\mathbb{R}^d$ contains $\mathbb{Z}^d$ as a uniformly locally finite metric lattice, and the latter has polynomial growth, so that $\mathbb{R}^d$ has polynomial growth of degree d.

The following is also a direct consequence of Proposition 3.36, and generalises Corollary 3.26(ii).

Corollary 3.41. *Let X be a uniformly coarsely proper and large-scale geodesic pseudo-metric space. Then X grows at most exponentially fast.*

Proof. From the assumptions and point (vii) of Proposition 3.36, we know that X is quasi-isometric to a connected graph of bounded degree. Such a graph has at most exponential growth, whence the conclusion. □

In this result, the assumption on the large-scale geodesicity of X cannot be dropped, as shown by Example 3.23(ii).

Since now growth is defined for a wider class of pseudo-metric spaces, it also makes sense to extend Proposition 3.24 to those spaces.

Proposition 3.42. *Let X, Y be two pseudo-metric spaces. Suppose Y is uniformly coarsely proper.*

(i) *If there is a coarse embedding $f\colon X \longrightarrow Y$, then X is uniformly coarsely proper. In particular, the growth type of X is well-defined.*

(ii) *If f is moreover large-scale Lipschitz, then $\beta_X \preccurlyeq \beta_Y$.*

(iii) *If f is moreover a quasi-isometry, then $\beta_X \simeq \beta_Y$.*

Proof. (i) By hypothesis, we have a subspace $Y_0 \subset Y$ and a surjective metric coarse equivalence $f\colon X \longrightarrow Y_0$. Let $g\colon Y_0 \longrightarrow X$ be a metric coarse equivalence so that f and g are inverses of each other in the metric coarse category. Let $c > 0$ be so that

$$d_X(g(y), g(y)) \geq 1$$

for all $y, y' \in Y_0$ with $d_Y(y, y') \geq c$. If now M_0 is a c–metric lattice in Y_0 and M is a c–metric lattice in Y containing M_0, then M is uniformly locally finite since Y is uniformly coarsely proper, and thus so is M_0. Hence $g(M_0)$ is a uniformly locally finite metric lattice in X, which implies that X is uniformly coarsely proper.

Points (ii) and (iii) follow from the corresponding points in Proposition 3.24 after passing to uniformly locally finite metric lattices in X and Y. □

As we proved, growth can be used to rule out the existence of a quasi-isometry between two uniformly coarsely proper pseudo-metric spaces.

Additionally, if these pseudo-metric spaces are large-scale geodesic, it can also rule out the existence of coarse embeddings.

Corollary 3.43. *Let X, Y be pseudo-metric spaces. Suppose that X is large-scale geodesic and that Y is uniformly coarsely proper.*

If there exists a coarse embedding from X to Y, then $\beta_X \preccurlyeq \beta_Y$.

Proof. Observe first that X is also uniformly coarsely proper by (i) of the previous result. Set $Y_0 := f(X)$. Let $c > 0$ be so that

$$d_Y(f(x), f(x')) \geq 1$$

for any $x, x' \in X$ with $d_X(x, x') \geq c$. By Proposition 2.25(i), f is in fact large-scale Lipschitz, *i.e.* there is $c_+ > 0$, $c'_+ \geq 0$ so that

$$d_Y(f(x), f(x')) \leq c_+ d_X(x, x') + c'_+$$

for any $x, x' \in X$. Now, fix $x_0 \in X$ and let L be a c−metric lattice in X containing x_0. Then $M_0 := f(L)$ is a 1−metric lattice in Y_0, so that there exists a 1−metric lattice M in Y containing M_0. Observe that $f_{|L}$ is injective and that

$$f(B_L(x_0, r)) \subset B_M(f(x_0), c_+ r + c'_+)$$

for any $r \geq 0$. As $f_{|L}$ is injective, it follows that $\beta_L^{x_0}(r) \leq \beta_M^{f(x_0)}(c_+ r + c'_+)$ for any $r \geq 0$, whence $\beta_L^{x_0} \preccurlyeq \beta_M^{f(x_0)}$. Hence $\beta_X \preccurlyeq \beta_Y$. □

Example 3.44. (i) As F_2 has exponential growth, it follows that any finitely generated containing a subgroup isomorphic to F_2 also has exponential growth. In particular, F_d has exponential growth for any $d \geq 1$.

(ii) For any $d \geq 1$, $\mathbb{Z}^d$ has polynomial growth of degree d while F_d has exponential growth. As $n^d \not\simeq e^n$ by Example 3.21(ii), it follows that $\mathbb{Z}^d$ is not quasi-isometric to F_d.

(iii) It follows from Corollary 3.43 that there does not exist any coarse embedding of F_2 into a euclidean space $\mathbb{R}^d$, $d \geq 1$. More generally, a regular tree of degree at least 3 does not coarsely embed into a euclidean space.

Let us close this part by presenting another way of defining growth for σ−compact locally compact groups, through their Haar measures.

Definition 3.45. Let G be a σ−compact locally compact group, μ a Haar measure on G, and d a measurable adapted pseudo-metric on G. For $r \geq 0$, we define the volume of the ball $B_d(e, r)$ as

$$\mathrm{Vol}(B_d(e, r)) := \int_{B_d(e,r)} d\mu$$

and the growth function of G with respect to d and μ as

$$v_{G,d,\mu}(r) := \mathrm{Vol}(B_d(e, r)), \ r \geq 0.$$

The growth type of G is then the equivalence class of $v_{G,d,\mu}$.

Observe that the growth type of G is independent of the choice of μ, since the latter is unique up to multiplicative constants.

Moreover, observe that if d' is another adapted pseudo-metric on G so that (G,d), (G,d') are quasi-isometric, then $v_{G,d,\mu} \simeq v_{G,d',\mu}$, so that the growth type of G is also independent of the choice of d.

We now prove this definition of growth is equivalent to the one above.

Proposition 3.46. *Let G be a σ−compact locally compact group, and let d, μ, $v_{G,d,\mu}$ be as above.*

(i) *The pseudo-metric space (G,d) is uniformly coarsely proper. In particular, its growth function β_G is well-defined.*

(ii) *The functions β_G and $v_{G,d,\mu}$ are equivalent.*

Proof. (i) Let $s > 0$ be large enough so that $B_d(e,s)$ is a neighborhood of e in G. Let $c > 2s$, and using Proposition 3.15, choose a c−metric lattice L in (G,d) containing e. We now show (L,d) is uniformly locally finite to conclude the proof of (i).

Let $r \geq 0$ and $\ell \in L$. Since $(\ell' C)_{\{\ell' \in L \,:\, d(\ell,\ell') \leq r\}}$ is a collection of pairwise disjoint balls all contained in $B_G(\ell, r+s)$, it follows that

$$\beta_L^{\ell}(r) v_{G,d,\mu}(s) \leq v_{G,d,\mu}(r+s) \tag{4}$$

and thus

$$\sup_{\ell \in L} \beta_L^{\ell}(r) \leq \frac{v_{G,d,\mu}(r+s)}{v_{G,d,\mu}(s)}$$

for any $r \geq 0$. Thus L is uniformly locally finite, and this proves (i).

(ii) Point (4) above already shows $\beta_L^{\ell} \preccurlyeq v_{G,d,\mu}$. On the other hand, if $R \geq \sup_{g \in G} d(g, L)$, then the balls $\ell B_G(e,1)$ cover G. It follows that

$$B_G(e,r) \subset \bigcup_{\ell \in L \cap B_G(e,r+R)} \ell B$$

for all $r \geq 0$, whence $v_{G,d,\mu}(r) \leq \beta_L^{e}(r+R) v_{G,d,\mu}(R)$ for all $r \geq 0$. Thus $v_{G,d,\mu} \preccurlyeq \beta_L^{\ell}$, and (ii) is proved. □

3.4 Growth of nilpotent groups

In this part, we generalise Corollary 3.29 and we prove that any finitely generated nilpotent group has polynomial growth.

Theorem 3.47. Any finitely generated nilpotent group has polynomial growth.

Proof. Let thus $G = \langle S \rangle$, $S = \{s_1, \dots, s_m\}$ be a finitely generated nilpotent group, of nilpotency class $c \geq 1$. We prove the theorem by induction on $c \geq 1$. The case $c = 1$ corresponds to a finitely generated abelian group, and is thus handled by Corollary 3.29.

Assume now that G has nilpotency class $c = 2$. This means that $[[G,G],G] = \{e_G\}$, and in particular $[G,G] \subset Z(G)$, *i.e.* $[G,G]$ is abelian. As it is also finitely generated by Proposition (from Chapter 1), we deduce from the base case that it has polynomial growth, say of degree $k \geq 1$. Now consider $g \in G$ of length n. Since G is generated by S, g is a product of elements of S, and we want to regroup powers of generators that are present in our original word and put them in a prescribed order. If $s, s' \in S$ are two generators so that ss' appears in g, then we write it as

$$ss' = s'ss^{-1}s'^{-1}ss' = [s', s]s's$$

and since $[s', s] \in Z(G)$, it commutes with all generators appearing before ss' in the writing of g, so that we can move $[s', s]$ at the left most end of g. After this operation, the product ss' in g has been replaced by $s's$. By repeating this operation, our word g can be written as

$$g = Cs_1^{k_1} \dots s_m^{k_m}$$

where $k_1, \dots, k_m \in \mathbb{N}$ and where C is a product of at most n^2 commutators of generators. Then

$$\beta_{(G,S)}(n) \leq n^{2k+m}$$

whence G has polynomial growth.

We handle the general case based on the same idea. Let $G = \langle s_1, \dots, s_m \rangle$ be finitely generated and nilpotent of nilpotency class $c \geq 1$. Then $[G,G]$ is nilpotent of nilpotency class $c-1$, and is finitely generated by Proposition 1.45. Thus, by the induction hypothesis, it has polynomial growth, say of degree k. If g has length n and ss' appears in the writing of g as a product of generators, we replace ss' by $[s', s]s's$. Now we must move $[s', s]$ at the left most end of g. If s'' is a generator so that $s''[s', s]$ appears in g, then we write

$$s''[s', s] = [[s', s], s''][s', s]s''$$

and $[[s', s], s''] \in [[G,G],G]$, which is finitely generated of nilpotency class $c-2$. In order to move all commutators that we get as we exchange generators in the initial word to the left most end of the word, we will have to perform at most n^3 exchanges of a commutator with a generator, and at each step we will get a double commutator that we will also have to move at the left most end of the word. Eventually this process will stop as c is finite. We conclude that there is $C \in \mathbb{N}$ so that

$$\beta_{(G,S)}(n) \leq n^{m+Ck}$$

and G has polynomial growth. □

For G a nilpotent group, the proof of the previous result does not give a precise formula computing the degree of polynomial growth, but it turns out such a formula actually exists. It is sometimes called the *Bass-Guivarch formula* [3, theorem 7.29].

Theorem 3.48. *Let G be a nilpotent group. For any $i \geq 1$, let r_i denote the torsion-free rank of the quotient $\gamma_i(G)/\gamma_{i+1}(G)$.*

Then G has polynomial growth of degree d, where $d := \sum_{i\geq 1} i r_i$.

For instance, applying this formula with the Heisenberg group $H(\mathbb{Z})$ which is nilpotent of class 2 shows that $H(\mathbb{Z})$ has polynomial growth of degree 4. In particular, $H(\mathbb{Z})$ and $\mathbb{Z}^3$ are not quasi-isometric, and $H(\mathbb{Z})$ does not coarsely embed into $\mathbb{Z}^3$.

Amazingly, the converse of Theorem 3.47 is also true, up to a subgroup of finite index. This is an outstanding result due to Gromov. A proof is far beyond the scope of this text, and can be found for instance in [3, theorem 12.1] or [6].

Theorem 3.49. *If G is a finitely generated group having polynomial growth, then G is virtually nilpotent.*

Here, recall that if $\mathcal{P}$ is a group property, a group is called *virtually $\mathcal{P}$* if G contains a finite index subgroup H having property $\mathcal{P}$.

3.5 Milnor's theorem

We saw above that on the one hand, all nilpotent groups have polynomial growth, and on the other hand all examples of exponential growth groups encountered so far contains non-abelian free subgroups. In particular, these examples are very far from being solvable. The goal of this section is to exhibit some examples of solvable groups (thus without any non-abelian free subgroups) that have exponential growth. This is accomplished through the following major result, due to John Milnor.

Theorem 3.50. *A finitely generated solvable group of subexponential growth is polycyclic.*

Proof. We claim that it is enough to prove that $[G,G]$ is finitely generated. Indeed, if this condition holds, then $[G,G]$ is a finitely generated solvable group of subexponential growth and, by induction on the derived length, it is polycyclic. On the other hand, $G/[G,G]$ is finitely generated and abelian, thus polycyclic as well. Hence G is polycyclic as an extension of two polycyclic groups (Proposition 1.47).

Let us then show that, under the assumptions of the theorem, $[G,G]$ is finitely generated. Since $G/[G,G]$ is finitely generated and abelian, we may find a sequence

$$G \geqslant H_s \geqslant \cdots \geqslant H_1 \geqslant H_0 = [G,G]$$

with $[G:H_s] < \infty$ and H_i/H_{i-1} is infinite cyclic for any $i = 1,\ldots,s$. Note that H_s is finitely generated by Corollary 3.7. Thus we can conclude that $[G,G]$ is finitely generated by applying iteratively the following claim.

Claim. *Let G be a finitely generated group of subexponential growth and suppose that H is a normal subgroup of G so that $G/H \cong \mathbb{Z}$. Then H is finitely generated.*

Proof of the claim. Let $a \in G$ be so that $G/H = \langle aH \rangle$, and let $X \subset G$ be a finite symmetric generating set of G. Without restrictions, we assume that $a \in X$. For any $x \in X$, there is $n \in \mathbb{Z}$ and $h \in H$ so that $x = a^n h$. Up to replacing any element $x \neq a^{\pm 1}$ by the corresponding h, we may assume that $X = \{a^{\pm 1}, h_1^{\pm 1}, \dots, h_\ell^{\pm 1}\}$ where $h_i \in H$ for all $i = 1, \dots, \ell$. Let $H^{(m)} \subset H$ denote the subgroup generated by the elements $a^j h_i^{\pm 1} a^{-j}$ with $i = 1, \dots, \ell$ and $j = 0, \dots, m$. Note that $H^{(0)} \subset H^{(1)} \subset H^{(2)} \subset \dots$ and let $H^+ := \bigcup_{m=0}^{\infty} H^{(m)}$. We prove that $H^+ = H^{(m)}$ for some $m \geq 1$. If not, for any $m \geq 0$, we can find $j_m \in \{1, \dots \ell\}$ so that $k_m := a^m h_{j_m} a^{-m} \in H^{(m)} \setminus H^{(m-1)}$. Now, for $m \in \mathbb{N}$, consider the products

$$k_0^{\varepsilon_0} k_1^{\varepsilon_1} \dots k_m^{\varepsilon_m}$$

where $\varepsilon_0, \dots, \varepsilon_m \in \{0, 1\}$. There are 2^{m+1} words of this type which represent disctinct group elements of length less than $3m+1$ with respect to X. Indeed the maximal length is attained when $\varepsilon_i = 1$ for all i so that the corresponding element is

$$k_0 \dots k_m = h_{j_0} a h_{j_1} a h_{j_2} \dots a h_{j_m} a^{-m}.$$

This implies that $\beta_{(G,X)}(3m+1) \geq 2^{m+1}$, contradicting the fact that G has subexponential growth. Thus $H^+ = H^{(m')}$ for some $m' \geq 1$. In the same way, exchanging the roles of a and a^{-1} we can show that the subgroup $H^- := \bigcup_{m=0}^{\infty} H^{(-m)}$, where $H^{(-m)}$ is the subgroup of G generated by the elements $a^{-j} h_i^{\pm 1} a^j$ with $i = 1, 2, \dots, \ell$ and $j = 0, \dots, m$, is finitely generated, say $H^- = H^{(-m'')}$ for some $m'' \geq 1$. It follows that

$$H = \Big\langle \bigcup_{j=-\infty}^{\infty} H^{(j)} \Big\rangle = \Big\langle \bigcup_{j=-m''}^{m'} H^{(j)} \Big\rangle$$

is finitely generated, and the claim is proved. □

As explained above, this concludes the proof of the theorem. □

Example 3.51. The lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is finitely generated (Proposition 1.43), solvable and not polycyclic (cf. Example 1.46). Thus, by Milnor's theorem, it has exponential growth. More generally, if A is finitely generated, solvable and that B is infinite finitely generated and solvable, then $A \wr B$ has exponential growth.

Another important result in growth theory, due to Wolf, is the following. A proof is presented for instance in [3, theorem 7.37].

Theorem 3.52. *A polycyclic group with subexponential growth is virtually nilpotent.*

Combining this with Milnor's theorem and Theorem 3.47, we deduce that intermediate growth cannot be observed among solvable groups.

Theorem 3.53. *A finitely generated solvable group either has exponential or polynomial growth. In the latter case, it is virtually nilpotent.*

4. Simple connectedness in the metric coarse category

In this part, we focus on another property that turns out to be invariant under metric coarse equivalence. The idea to keep in mind is that this construction is a coarse analog of the construction of the fundamental group of a topological space.

4.1 Coarsely simply connected pseudo-metric spaces

Definition 4.1. Let (X, d_X) be a pseudo-metric space and let $c > 0$. Two c−paths $\xi = (x_0, \dots, x_m)$, $\eta = (y_0, \dots, y_n)$ in X are c−elementarily homotopic if $x_0 = y_0$, $x_m = y_n$ and if either

$$n = m + 1 \text{ and } (y_0, \dots, y_n) = (x_0, \dots, x_i, y_{i+1}, x_{i+1}, \dots, x_m)$$

or

$$m = n + 1 \text{ and } (x_0, \dots, x_m) = (y_0, \dots, y_i, x_{i+1}, y_{i+1}, \dots, y_m)$$

for some index i.

Additionally, we say that two c−paths ξ, η are c−*homotopic* if there exists a sequence $\xi_0 = \xi, \xi_1, \dots, \xi_\ell = \eta$ of c−paths so that ξ_{j-1}, ξ_j are c−elementarily homotopic for any $j = 1, \dots, \ell$.

For $x_0 \in X$, a c−*loop* in X at x_0 is a c−path that starts and ends at x_0.

Here is a first example of homotopic paths.

Lemma 4.2. Let (X, d_X) be a pseudo-metric space, $c > 0$, and let $\xi = (x_0, \dots, x_n)$, $\eta = (y_0, \dots, y_n)$ be two c−paths in X so that $x_0 = y_0$, $x_n = y_n$.

If $d_X(x_i, y_i) \leq c$ for any $i = 1, \dots, n$, then ξ and η are $2c$−homotopoic.

Proof. The idea here is to start from the sequence ξ and progressively introduce the y_i's and delete the x_i's, alternating each of these moves. Explicitly, we set $\xi_0 := \xi$, and then

$$\begin{aligned} \xi_1 &:= (x_0, y_1, x_1, \dots, x_n) \\ \xi_2 &:= (x_0, y_1, x_2, \dots, x_n) \\ \xi_3 &:= (x_0, y_1, y_2, x_2, \dots, x_n) \end{aligned}$$

and so on until reaching $\xi_{2n-2} := (x_0 = y_0, y_1, \dots, y_n) = \eta$. The sequence $\xi_0, \dots, \xi_{2n-2}$ is then a sequence of $2c$−paths so that ξ_{j-1}, ξ_j are $2c$−elementarily homotopic for any $j = 1, \dots, 2n-2$. Thus ξ and η are $2c$−homotopic. □

Definition 4.3. Let X be a pseudo-metric space and $x_0 \in X$. If $c'' \geq c' > 0$, we say that X has the Property SC(c', c'') if any c'−loop in X at x_0 is c''−homotopic to the trivial loop (x_0).

Remark 4.4. For constants $c'' \geq c' \geq c > 0$, a c−coarsely connected pseudo-metric space X has Property SC(c', c'') for one choice of base point in X if and only if it has Property SC(c', c'') for any other choice of base point.

Thus, when irrelevant, we do not specify the choice of the base point.

The following result will be useful later.

Lemma 4.5. Let $c'' \geq c' \geq c > 0$, and suppose X is $c-$geodesic. Let $x_0 \in X$.

If X has Property SC(c', c''), then it also has Property SC(c'', c'').

Proof. Let $\xi = (x_0, x_1, \dots, x_n = x_0)$ be a $c''-$loop at x_0 in X. Using $c-$geodesicity, we can insert new points in between those of ξ in order to get a $c'-$loop $\eta = (x_0, y_1, \dots, y_m, x_0)$ based at x_0 in X that is $c''-$homotopic to ξ. As X has SC(c', c''), there is a $c''-$homotopy from η to (x_0), and thus ξ is also $c''-$homotopic to (x_0). Hence X has SC(c'', c''). □

We now turn to coarse simple connectedness.

Definition 4.6. Let (X, d_X) be a pseudo-metric space, and let $c > 0$. We say that X is $c-$coarsely simply connected if it is $c-$coarsely connected and if, for any $c' \geq c$, there exists $c'' \geq c'$ so that X has Property SC(c', c'').

We say that X is *coarsely simply connected* if it is $c-$coarsely simply connected for some $c > 0$.

Remark 4.7. If $C \geq c > 0$ and if X is $c-$coarsely connected, then X is $c-$coarsely simply connected if and only if it is $C-$coarsely connected.

We can thus state the main result of this section.

Theorem 4.8. Coarse simple connectedness is invariant under metric coarse equivalence.

Proof. Let $f\colon (X, d_X) \longrightarrow (Y, d_Y)$ be a metric coarse equivalence between two pseudo-metric spaces, and let $c := \sup_{y \in Y} d_Y(y, f(X)) < \infty$. Without restriction, we may assume that X is $c-$coarsely simply connected. In particular, X is $c-$coarsely connected, and from Proposition 2.24, we know that Y is $C-$coarsely connected for some $C > 0$. Let $L := \max(c, C)$. We claim that Y is $L-$coarsely simply connected. Let $\ell' \geq L$. We are going to prove there exists $\ell'' \geq \ell'$ so that Y has Property SC(ℓ', ℓ'').

To that aim, fix $y_0 \in Y$ and $\xi = (y_0, y_1, \dots, y_{n-1}, y_n = y_0)$ a $\ell'-$loop in Y based at y_0. By Remark 4.4, up to changing the base point, we may assume that y_0 is in the image of f, and write $y_0 = f(x_0)$ for some $x_0 \in X$. For each $1 \leq i \leq n-1$, let $x_i \in X$ be so that $d_Y(y_i, f(x_i)) \leq c$, and set $x_n := x_0$. Then it follows that

$$d_Y(f(x_{i-1}), f(x_i)) \leq d_Y(f(x_{i-1}), y_{i-1}) + d_Y(y_{i-1}, y_i) + d_Y(y_i, f(x_i)) \leq 2c + \ell' \tag{5}$$

for any $1 \leq i \leq n$. On the other hand, as f is coarsely expansive, there is $s \geq 0$ so that

$$d_X(x, x') \geq s \Longrightarrow d_Y(f(x), f(x')) \geq 2c + \ell' + 1.$$

We deduce from this implication and from (5) that $d_X(x_{i-1}, x_i) \leq s$ for any $1 \leq i \leq n$, in other words $(x_0, x_1, \dots, x_{n-1}, x_n = x_0)$ is a $s-$loop based at x_0 in X. In particular,

$(x_0, x_1, \dots, x_{n-1}, x_n = x_0)$ is a $\max(s, c)$–loop based at x_0 in X, and X being c–coarsely simply connected, we deduce there is a constant $c'' \geq s$ so that $(x_0, x_1, \dots, x_{n-1}, x_n = x_0)$ is c''–homotopic to (x_0). Unraveling the definition, we find a sequence $\xi_0, \xi_1, \dots, \xi_r$ of c''–loops so that

$$\xi_0 = (x_0, x_1, \dots, x_n = x_0),\ \xi_r = (x_0)$$

and ξ_{j-1}, ξ_j are c''–elementarily homotopic for any $1 \leq j \leq r$. Now, using that f is coarsely Lipschitz, we find $\ell_1'' > 0$ so that

$$d_X(x, x') \leq c'' \Longrightarrow d_Y(f(x), f(x')) \leq \ell_1''.$$

For any $0 \leq j \leq r$, set $\eta_j := f(\xi_j)$. Combining the above inequality and the fact that ξ_j is a c''–loop at x_0 in X, we deduce that η_j is a ℓ_1''–loop at y_0 in Y for all $0 \leq j \leq r$, and moreover $\eta_r = (f(x_0)) = (y_0)$ is the constant loop at y_0. Also, for any $1 \leq j \leq r$, η_{j-1}, η_j are ℓ_1''–elementarily homotopic. We thus conclude that η_0 and (y_0) are ℓ_1''–homotopic.

To conclude, it remains to notice that

$$\xi = (y_0, y_1, \dots, y_{n-1}, y_n = y_0),\ \eta_0 = (y_0, f(x_1), \dots, f(x_{n-1}), f(x_n) = y_0)$$

are both $\max(\ell', \ell_1'')$–loops at y_0 and that $d_Y(y_i, f(x_i)) \leq c$ for any $0 \leq i \leq n$. Hence Lemma 4.2 ensures that ξ and η_0 are $2\max(c, \max(\ell', \ell_1''))$–homotopic. Thus we deduce that ξ and (y_0) are $2\max(c, \max(\ell', \ell_1''))$–homotopic, and setting $\ell'' := 2\max(c, \max(\ell', \ell_1''))$, it follows that Y has Property SC(ℓ', ℓ''). We conclude that Y is L–coarsely simply connected, as claimed. ◻

Example 4.9. Since the one-point space is coarsely simply connected, it follows from Example 2.13(i) and the previous result that any pseudo-metric space of finite diameter is coarsely simply connected.

Here is a general result giving additional examples, such as the euclidean space $\mathbb{R}^n$ for $n \geq 1$ or the unit sphere S^n for any $n \geq 2$.

Proposition 4.10. Let X be a geodesic metric space.

If X is simply connected, then X is coarsely simply connected.

Proof. Let $c' > 0$, $x_0 \in X$ and $\xi = (x_0, x_1, \dots, x_n = x_0)$ a c'–loop based at $x_0 \in X$. Let $L := \sum_{i=1}^{n} d(x_{i-1}, x_n)$. As X is geodesic, we may pick a continuous loop $\varphi\colon [0, L] \longrightarrow X$ and a sequence of real numbers $(s_i)_{0 \leq i \leq n}$ so that $0 = s_0 \leq s_1 \leq \cdots \leq s_n = L$, $\varphi(s_i) = x_i$ for any $i = 0, \dots, n$ and

$$d(x_{i-1}, x_i) = d(\varphi(s_{i-1}), \varphi(s_i)) = |s_{i-1} - s_i| \leq c'$$

for any $i = 0, \dots, n$. Using now simple connectedness, there is a continuous homotopy $H\colon [0, L] \times [0, 1] \longrightarrow X$ so that

$$\forall s \in [0, L],\ H(s, 0) = \varphi(s),\ H(s, 1) = x_0$$

and

$$\forall t \in [0, 1],\ H(0, t) = H(1, t) = x_0.$$

Since H is continuous on the compact space $[0,L]\times[0,1]$, it is uniformly continuous, so we may pick $N \geq 1$ so that

$$d(H(s,t),H(s',t')) \leq c'$$

whenever $|s-s'| \leq \frac{L}{N}$ and $|t-t'| \leq \frac{1}{N}$. Hence there is a subsequence $(r_h)_{0\leq h\leq M}$ of $(s_i)_{0\leq i\leq n}$ so that $0 = r_0 \leq r_1 \leq \cdots \leq r_h = L$ and $|r_h - r_{h-1}| \leq \frac{L}{N}$ for $1 \leq h \leq M$. For any $j \in \{0,\dots,N\}$, we then set

$$\xi_j = \big(H\big(r_h, \frac{j}{N}\big)\big)_{0\leq h\leq M}$$

so that in particular ξ and ξ_0 are c'–homotopic. By Lemma 4.2, the loops ξ_0 and ξ_N are $2c'$–homotopic. Since $\xi_N = (x_0)$, we conclude that ξ is c''–homotopic to (x_0) for some $c'' \geq c'$, concluding the proof. □

Proposition 4.11. Let (Y,d_Y) be a pseudo-metric space and $Z \subset Y$ a coarse retract.

If Y is coarsely simply connected, then so is Z.

Proof. Let $r\colon Y \longrightarrow Z$ be a coarse retraction. We may assume that there is $z_0 \in Z$ so that $r(z_0) = z_0$. Let Φ be an upper control for r, *i.e.*

$$d_Z(r(y),r(y')) \leq \Phi(d_Y(y,y'))$$

for any $y,y' \in Y$. Let also $K \geq 0$ be so that $d_Z(z,r(z)) \leq K$ for all $z \in Z$. Choose a constant $c > 0$ so that Y is c–coarsely connected and let $c' \geq c$. Consider a $\Phi(c')$–loop η in Z at z_0. By hypothesis on Y, there is $k'' \geq \Phi(c')$ and a sequence $\xi_0 = \eta, \xi_1, \dots, \xi_\ell = (z_0)$ of k''–loops in Y at z_0 so that ξ_{j-1},ξ_j are k''–elementarily homotopic for $j = 1,\dots,\ell$. Then

$$r(\xi_0) = \eta, r(\xi_1), \dots, r(\xi_\ell) = (z_0)$$

is a sequence of $\Phi(k'')$–loops in Z at z_0 so that $r(\xi_{j-1}), r(\xi_j)$ are $\Phi(k'')$–elementarily homotopic for $j = 1,\dots,\ell$. Hence Z has Property SC($\Phi(c'),\Phi(k'')$), so that it is coarsely simply connected. □

4.2 Combinatorial homotopy

If X is a simplicial complex, we denote $X^0 \subset X^1 \subset X^2 \subset \dots$ the nested sequence of its squeletons, and X_{top} its topological realisation, the Hausdorff topological space obtained from X^0 by attaching cells of dimension $1,2,3,\dots$. Note that a graph is a one dimensional simplicial complex.

Definition 4.12. Let X be a simplicial complex. A combinatorial path in X from a vertex $x \in X^0$ to a vertex $y \in X^0$ is a sequence of oriented edges

$$\xi = ((x_0,x_1),\dots,(x_{m-1},x_m))$$

with $x_0 = x$ and $x_m = y$. Such a path is denoted $\xi = (x_0,x_1,\dots,x_m)$.

If $\xi = (x_0, x_1, \dots, x_m)$ is a combinatorial path, then its *inverse path* is the path $\xi^{-1} := (x_m, x_{m-1}, \dots, x_0)$. The *product* of two combinatorial paths $\xi = (x_0, x_1, \dots, x_m)$, $\eta = (y_0, y_1, \dots, y_n)$, denoted $\xi\eta$, is defined when $x_m = y_0$ and is given by

$$\xi\eta = (x_0, x_1, \dots, x_m, y_1, y_2, \dots, y_n).$$

A *combinatorial loop* in X based at $x_0 \in X$ is a combinatorial path from x_0 to x_0.

We can define homotopies between paths in simplicial complexes.

Definition 4.13. Let X be a simplicial complex and let x, y be two vertices of X. Two combinatorial paths from x to y are elementarily graph homotopic if they are of the form

$$(x_0, x_1, \dots, x_n), (x_0, x_1, \dots, x_i, u, x_i, \dots, x_n)$$

with $x_0 = x$, $x_n = y$, where (x_i, u) is an oriented edge of X.

Additionally, two combinatorial paths ξ, ξ' in X from x to y are *graph homotopic* if there is a sequence $\xi_0 = \xi, \xi_1, \dots, \xi_\ell = \xi'$ of combinatorial paths so that ξ_{j-1}, ξ_j are graph elementarily homotopic for any $j \in \{1, \dots, \ell\}$.

Also, we say that ξ, ξ' are *triangle homotopic* if they are of the form

$$\xi = (x_0, \dots, x_n), \xi' = (x_0, \dots, x_i, u, x_{i+1}, \dots, x_n)$$

where $\{x_i, u, x_{i+1}\}$ is a 2–simplex in X.

Lastly, two combinatorial paths ξ, ξ' from x to y are *combinatorially homotopic* if there exists a sequence $\xi_0 = \xi, \xi_1, \dots, \xi_\ell = \xi'$ of combinatorial paths from x to y so that ξ_{j-1}, ξ_j are either elementarily graph homotopic or triangle homotopic for any $j \in \{1, \dots, \ell\}$.

Combinatorial homotopy between combinatorial paths is an equivalence relation that is compatible with products and inverses, in the sense that if ξ, ξ', η, η' are combinatorial paths so that ξ, η (resp. ξ', η') are combinatorially homotopic and so that $\xi\eta$ is defined, then ξ^{-1}, η^{-1} (resp. ξ'^{-1}, η'^{-1}) are combinatorially homotopic, and $\xi'\eta'$ is defined and combinatorially homotopic to $\xi\eta$.

Definition 4.14. Let X be a simplicial complex. Let $\xi = (x_0, \dots, x_n)$ be a combinatorial path in X. The topological realisation of ξ is a continuous path $\xi_{\text{top}} : I \longrightarrow X$ with origin x_0 and end x_n, where

$$I = [t_0, t_n] = \bigcup_{j=1}^{n} [t_{j-1}, t_j]$$

is an interval of the real line made up of n subintervals with disjoint interiors, and ξ_{top} maps successively $[t_0, t_1]$ onto the edge of ξ from x_0 to x_1, $[t_1, t_2]$ onto the edge of ξ from x_1 to x_2,..., and $[t_{n-1}, t_n]$ onto the edge of ξ from x_{n-1} to x_n.

The following lemma is an immediate consequence of the definitions.

Lemma 4.15. (i) Any loop in X_{top} based at x_0 is homotopic to the topological realisation of a combinatorial loop based at x_0.

(ii) Let ξ, ξ' be combinatorial paths. Then ξ, ξ' are combinatorially homotopic if and only if $\xi_{\text{top}}, \xi'_{\text{top}}$ are homotopic in the topological sense.

Lemma 4.16. Let X be a connected simplicial complex and let $\xi = (x_0, x_1, \ldots, x_n = x_0)$ be a combinatorial path in X based at $x_0 \in X^0$. Suppose ξ_{top} is homotopically trivial as a loop in X_{top} based at x_0.

Then ξ is combinatorially homotopic to a product

$$\prod_{j=1}^{N} u_j r_j u_j^{-1}$$

where, for all $j \in \{1, \ldots, N\}$, u_j is a combinatorial path from x_0 to some vertex $z_j \in X^0$, and r_j has length 3 and all its vertices belong to a common $2-$simplex, *i.e.* r_j is of the form (z_j, z'_j, z''_j, z_j).

Proof. First, observe that if

$$\eta = (x_0, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{k-1}, x_0),\ \eta' = (x_0, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{k-1}, x_0)$$

are two triangle homotopic combinatorial loops based at x_0 where $\{y_{i-1}, y_i, y_{i+1}\}$ is a $2-$simplex in X, then setting

$$u := (x_0, \ldots, y_{i-1}, y_{i+1}),\ r := (y_{i+1}, y_{i-1}, y_i, y_{i+1})$$

produces an elementary graph homotopy between η and $uru^{-1}\eta'$, and similarly η' and $ur^{-1}u^{-1}\eta$ are elementarily graph homotopic.

Now, fix a combinatorial path ξ as in the statement. By assumption and the previous lemma, ξ and (x_0) are combinatorially homotopic, so that there is a sequence $\xi_0 = \xi, \xi_1, \ldots, \xi_\ell = (x_0)$ of combinatorial loops at x_0 with ξ_{j-1}, ξ_j being either elementarily graph homotopic or triangle homotopic, for any $j \in \{1, \ldots, \ell\}$. Say ξ_{j-1}, ξ_j are triangle homotopic for N of the j's. Applying N times the observation above written for η, η', the conclusion follows. $\square$

For our next purposes, we will take for granted the next proposition.

Proposition 4.17. Let X be a connected simplicial complex, and let d_1 be the combinatorial metric on the geometric realisation Y of X^1.

On the topological realisation Z of X^2, there exists a unique combinatorial metric d_2 making each edge an interval of length 1, each $2-$cell of Z a euclidean equilateral triangle of side-length 1, and such that (Z, d_2) is a complete geodesic space. Moreover, the inclusion $(Y, d_1) \hookrightarrow (Z, d_2)$ is a quasi-isometry.

4.3 The Rips $2-$complex of a pseudo-metric space

In this part, we show that the coarse simple connectedness of a pseudo-metric space is encoded in the simple connectedness of a topological space associated to the initial space.

Definition 4.18. Let (X,d) be a pseudo-metric space and $c > 0$. The Rips simplicial 2−complex $\mathrm{Rips}^2_c(X,d)$ is the 2−dimensional simplicial complex with X as set of vertices, pairs (x,y) of distinct points of X with $d(x,y) \leq c$ as set of oriented edges, and triples (x,y,z) of distinct points of X with mutual distances bounded by c as set of oriented 2−simplices.

The Rips 2−complex is the geometric realisation of this 2−complex and is also denoted $\mathrm{Rips}^2_c(X,d)$. It is endowed with the combinatorial metric of Proposition 4.17.

Observe that if $c'' \geq c' > 0$, there is a canonical inclusion

$$j\colon \mathrm{Rips}^2_{c'}(X,d) \hookrightarrow \mathrm{Rips}^2_{c''}(X,d)$$

which is the identity on the 0−skeletons.

The next result is also an immediate consequences of the definitions.

Proposition 4.19. *Let (X,d) be a pseudo-metric space, $x_0 \in X$, and $c'' \geq c' \geq c > 0$. Then the following claims hold.*

(i) *X is c−coarsely connected if and only if $\mathrm{Rips}^2_c(X,d)$ is connected.*

(ii) *X is c−coarsely geodesic if and only if the natural inclusion $X \hookrightarrow \mathrm{Rips}^2_c(X,d)$ is a metric coarse equivalence.*

(iii) *X is c−large-scale geodesic if and only if the natural inclusion $X \hookrightarrow \mathrm{Rips}^2_c(X,d)$ is a quasi-isometry.*

(iv) *If $\mathrm{Rips}^2_{c'}(X,d)$ is connected, then $\mathrm{Rips}^2_{c''}(X,d)$ is connected.*

(v) *X has Property $\mathrm{SC}(c',c'')$ if and only if the induced homomorphism*

$$j_*\colon \pi_1(\mathrm{Rips}^2_{c'}(X,d)) \longrightarrow \pi_1(\mathrm{Rips}^2_{c''}(X,d))$$

is trivial.

Proof. (iv) follows from (i) and the fact that c'−coarse connectedness implies c''−coarse connectedness.

Let us prove (v). Assume first that X has $\mathrm{SC}(c',c'')$. Let $\gamma \in \pi_1(\mathrm{Rips}^2_{c'}(X,d))$. By Lemma 4.15(i), γ can be represented by a combinatorial loop at x_0 in X, and the latter defines a c'−loop ξ in X based at x_0. As X has $\mathrm{SC}(c',c'')$, ξ is c''−homotopic to (x_0). This homotopy provides a combinatorial homotopy from ξ, viewed as a combinatorial loop in $\mathrm{Rips}^2_{c''}(X,d)$, to the trivial loop. Hence $j_*(\gamma) = 1 \in \pi_1(\mathrm{Rips}^2_{c''}(X,d))$.

Conversely, if j_* has trivial image and ξ is a c'−loop in X at x_0, then there is a homotopy from ξ_{top} viewed as a loop in $\mathrm{Rips}^2_{c''}(X,d)$ to the trivial loop. From Lemma 4.15(ii), there is a combinatorial homotopy from ξ, viewed as a combinatorial loop in $\mathrm{Rips}^2_{c''}(X,d)$, to the constant loop. The latter combinatorial homotopy is a c''−homotopy from ξ to the constant loop (x_0). Thus X has Property $\mathrm{SC}(c',c'')$. □

Therefore, the coarse simple connectedness of a space (X,d) is equivalent to the simple connectedness of its Rips $2-$complex.

Proposition 4.20. *Let (X,d) be $c-$geodesic. The following claims are equivalent.*

(i) *X is $c-$coarsely simply connected.*

(ii) *There exists $k \geq c$ so that* $\mathrm{Rips}_k^2(X,d)$ *is simply connected.*

(iii) *There exists $k \geq c$ so that* $\mathrm{Rips}_K^2(X,d)$ *is simply connected for any $K \geq k$.*

Proof. (i) $\Longrightarrow$ (iii) : Suppose that X is $c-$coarsely simply connected, *i.e.* for any $c' \geq c$ there exists $c'' \geq c$ so that X has SC(c', c''). In particular, there is $k \geq c$ so that X has Property SC(c, k). Let $K \geq k$. By Lemma 4.5, that we may apply since X is $c-$geodesic, X also has SC(k, k), and thus also SC(K, K). It follows from Proposition 4.19(v) that $\mathrm{Rips}_K^2(X,d)$ is simply connected.
(iii) $\Longrightarrow$ (ii) is obvious.
(ii) $\Longrightarrow$ (i) : Assume that $\mathrm{Rips}_k^2(X,d)$ is simply connected for some $k \geq c$. Since it is also a geodesic space, it is coarsely simply connected by Proposition 4.10. As there is a metric coarse equivalence between X and $\mathrm{Rips}_k^2(X,d)$ by Proposition 4.19(ii), and as coarse simple connected is invariant under metric coarse equivalence by Theorem 4.8, we conclude that X is a coarsely simply connected. □

Let us conclude this part by proving another caracterisation of coarse simple connectedness.

Proposition 4.21. *Let X be coarsely geodesic. The following claims are equivalent.*

(i) *The space X is coarsely simply connected.*

(ii) *The space X is coarsely equivalent to a simply connected geodesic metric space.*

Proof. Assume first that X is coarsely simply connected. As it is coarsely geodesic, it is coarsely equivalent to a geodesic metric space Z. Then Z is also coarsely simply connected by Theorem 4.8, so $\mathrm{Rips}_c^2(Z)$ is simply connected for $c > 0$ large enough. As Z and $\mathrm{Rips}_c^2(Z)$ are coarsely equivalent, it follows that X and $\mathrm{Rips}_c^2(Z)$ are coarsely equivalent as well.

Conversely, if $f : X \longrightarrow Y$ is a metric coarse equivalence and Y is geodesic and simply connected, then Y is coarsely simply connected by Proposition 4.10, whence X is coarsely simply connected as well by Theorem 4.8. □

4.4 Bounded presentations

Recall from Chapter 1 that a group G is generated by a set S if there is a surjective homomorphism $\pi : F_S \longrightarrow G$, where F_S is the free group on S. The relations of such a generation are the elements of $\mathrm{Ker}(\pi)$. The set S is often called an *alphabet*.

Definition 4.22. A presentation of a group G is a triple (S,π,R), where (S,π) is a generation of G and R is a subset of F_S generating $\mathrm{Ker}(\pi)$ as a normal subgroup. When given such a data, we write

$$G = \langle S \mid R\rangle.$$

If (S,π,R) is a presentation of G, the subset $R \subset F_S$ is called a *relating subset*, and its elements are the *relators* of the presentation. Observe that the relations are the elements of the form

$$\prod_{i=1}^{k} w_i r_i w_i^{-1}$$

with $k \geq 0$, $r_1,\dots,r_k \in R \cup R^{-1}$, $w_1,\dots,w_k \in F_S$.

Definition 4.23. A bounded presentation for a group G is a presentation $G = \langle S \mid R\rangle$ with R a set of relators of bounded length.

If $G = \langle S \mid R\rangle$ is a bounded presentation for G, we say that G is *boundedly presented over* S.

If G has a presentation $\langle S \mid R\rangle$ with S and R finite, we say that G is *finitely presented*, and if G is a topological group with a bounded presentation $\langle S \mid R\rangle$ and with S being compact, we say that G is *compactly presented*. Observe that any finitely (resp. compactly) presented group is finitely (resp. compactly) generated.

Example 4.24. (i) For any $n \geq 1$, the non-abelian free group $F_n = \langle a_1,\dots,a_n \mid \emptyset\rangle$ is finitely presented.
(ii) The group $\mathbb{Z}^2 = \langle a,b \mid [a,b]\rangle$ is finitely presented. In fact, as we will see below, any finitely generated polycyclic group is finitely presented.
(iii) It is a well known fact that if $G = \langle S_G \mid R_G\rangle$ and $H = \langle S_H \mid R_H\rangle$, then $G * H = \langle S_G \cup S_H \mid R_G \cup R_H\rangle$.

Lemma 4.25. *Let*

$$1 \longrightarrow N \xrightarrow{\iota} H \xrightarrow{\pi} Q \longrightarrow 1$$

be a short exact sequence of groups.

(i) *Assume that G is boundedly generated over a set S and that is generated as a normal subgroup by $N \cap \overline{S}^n$ for some n. Then Q is boundedly presented over $\pi(S)$.*
(ii) *Let $\rho\colon F_S \twoheadrightarrow G$ be a generation of G so that the kernel of $\pi\circ\rho\colon F_S \twoheadrightarrow Q$ is generated as a normal subgroup by a set R of relators of length at most $k \in \mathbb{N}$. Then N is generated as a normal subgroup of G by $\rho(\overline{S}^k) \cap N$.*

Proof. (i) For each $r \in R$, let $\underline{r}$ denote the word in the letters of $\pi(\overline{S})$ obtained by replacing each letter $s \in \overline{S}$ of r by the corresponding letter $\pi(s) \in \pi(\overline{S})$, and let $\underline{R_1}$ denote the set of those $\underline{r}$. For each $g \in N \cap \overline{S}^n$, choose $s_1,\dots,s_n \in \overline{S}$ so that $g = s_1\dots s_n$, and let $\underline{R_2}$ denote the set of words of the form $\pi(s_1)\dots\pi(s_n)$. Then

$$\langle \pi(\overline{S}) \mid \underline{R_1} \cup \underline{R_2}\rangle$$

is a bounded presentation of G/N.

(ii) Let M be the normal subgroup of G generated by $\rho(S^k)\cap N$. It is clear that $M \subset N$, and we must show that $M = N$. Upon replacing G by G/M, we may assume that $M = \{e\}$, and we show that $N = \{e\}$. Clearly, one has $\mathrm{Ker}(\rho) \subset \rho(\pi \circ \rho)$. Let $r \in R$, viewed as a word in the letters of $S \cup S^{-1}$. We have $\rho(r) \in \overline{S}^k \cap N$, and therefore $r \in \mathrm{Ker}(\rho)$. Since R generates $\mathrm{Ker}(\pi\circ\rho)$ as a normal subgroup of F_S, it follows that $\mathrm{Ker}(\pi\circ\rho) \subset \mathrm{Ker}(\rho)$. Hence $N = \{e\}$. □

Lemma 4.26. *Let*

$$1 \longrightarrow N \xrightarrow{\iota} H \xrightarrow{\pi} Q \longrightarrow 1$$

be a short exact sequence of groups. Assume that N, Q are boundedly presented, i.e.

$$N = \langle S_N \mid R_N\rangle,\ Q = \langle S_Q \mid R_Q\rangle$$

with S_N (resp. S_Q) being symmetric and containing e_N (resp. e_Q), and so that

$$m_N := \sup_{r\in R_N} |r|_{S_N} < \infty,\ m_Q := \sup_{r\in R_Q} |r|_{S_Q} < \infty.$$

Let $S'_G \subset G$ be symmetric, containing e_G and so that $\pi(S'_G) = S_Q$. Let $\sigma\colon S_Q \longrightarrow S'_G$ be so that $\pi(\sigma(s)) = s$ for any $s \in S_Q$. Assume furthermore that there exist $k, \ell \geq 1$ so that

$$(S'_G S_N S'_G) \cap N \subset (S_N)^k,\ (S'_G)^{m_Q} \cap N \subset (S_N)^\ell.$$

Then G is boundedly presented, i.e. there exists a set R_G of words of bounded length in the letters of $S_N \cup S'_G$ so that

$$G = \langle S_N \cup S'_G \mid R_G\rangle$$

is a bounded presentation for G.

Proof. See [5, lemma 7.A.12]. □

In particular, for locally compact groups we get the following statement.

Proposition 4.27. *Let*

$$1 \longrightarrow N \xrightarrow{\iota} H \xrightarrow{\pi} Q \longrightarrow 1$$

be a short exact sequence of locally compact groups, where the topology of N coincides with the topology induced by ι, and π is continuous and open. Assume that N, Q are compactly presented, i.e.

$$N = \langle S_N \mid R_N\rangle,\ Q = \langle S_Q \mid R_Q\rangle$$

with $S_N \subset N$ (resp. $S_Q \subset Q$) compact, and so that

$$m_N := \sup_{r\in R_N} |r|_{S_N} < \infty,\ m_Q := \sup_{r\in R_Q} |r|_{S_Q} < \infty.$$

Then the group G has a presentation $\langle S_G \mid R_G\rangle$ with $S_G \subset G$ compact and

$$\sup_{r\in R_G} |r|_{S_G} < \infty.$$

Proof. Keeping the notations from the previous statement, S'_G can be chosen compact by Lemma 1.39. Then $S_G = S_N \cup S'_G$ is a compact generating set for G, and the lengths of the relators in R_G are bounded by $\max(m_N, k+3, m_Q+\ell)$. □

We conclude this part relating bounded presentations and coarse simple connectedness.

Theorem 4.28. *Let G be a group with a generating set S. The following claims are equivalent.*

(i) *The group G is boundedly presented over S.*

(ii) $\mathrm{Rips}^2_c(G, d_S)$ *is simply connected for some* $c > 0$.

(iii) $\mathrm{Rips}^2_c(G, d_S)$ *is simply connected for all* $c > 0$ *large enough.*

(iv) *The metric space (G, d_S) is coarsely simply connected.*

Proof. The equivalences (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) have already been showed in Proposition 4.20.

(i) $\Longrightarrow$ (ii) : Let $G = \langle S \mid R\rangle$ be a bounded presentation, and set $m := \max_{r\in R} \ell_S(r)$. Let $c \geq \max(1, \frac{m}{2})$, and let ξ be a loop based at e_G in the topological realisation of $\mathrm{Rips}^2_c(G, d_S)$. By Lemma 4.15(i), we can assume that ξ is the topological realisation of a combinatorial loop

$$\eta = (e_G, s_1, s_1s_2, \dots, s_1\dots s_{k-1}, s_1\dots s_{k-1}s_k = e_G)$$

with $s_1,\dots,s_k \in \overline{S}$. There are relators $r_1,\dots,r_\ell \in R\cup R^{-1}$ and words $w_1,\dots,w_\ell \in F_S$ so that

$$s_1\dots s_k = \prod_{j=1}^{\ell} w_j r_j w_j^{-1}.$$

Let $j \in \{1,\dots,\ell\}$. As $\ell_S(r_j) \leq m$, any triple of vertices of r_j is in a common 2−simplex, so the prefix of the word $w_j r_j w_j^{-1}$ constitute a combinatorial loop that is combinatorially homotopic to the constant loop. Hence η is combinatorially homotopic to the constant loop, and thus ξ is homotopic to the trivial loop. It follows that $\mathrm{Rips}^2_c(G, d_S)$ is simply connected.

(iii) $\Longrightarrow$ (i) : Let $m \geq 1$ be an integer so that $\mathrm{Rips}^2_m(G, d_S)$ is simply connected. Let $\pi\colon F_S \longrightarrow G$ be a surjective morphism, and write $N = \mathrm{Ker}(\pi)$. Let $w \in N$, and write $w = s_1\dots s_k \in \overline{S}$. Consider

$$\eta = (e_G, s_1, \dots, s_1\dots s_{k-1}, s_1\dots s_k = e_G)$$

which is a combinatorial loop based at e_G in $\mathrm{Rips}^2_m(G, d_S)$. Then η is combinatorially homotopic to some combinatorial loop

$$\prod_{j=1}^{N} u_j r_j u_j^{-1}$$

as in Lemma 4.16, where each r_j is a combinatorial loop of length at most 3 in $\mathrm{Rips}^2_m(G, d_S)$. Letting R denote the set of these r_j, it follows that $G = \langle S \mid R\rangle$. □

4.5 Compactly presented groups

The first result of this section is an immediate consequence of our previous observations.

Theorem 4.29. *Let $G = \langle S \rangle$ be a compactly generated locally compact group. Then G is compactly presented if and only if (G, d_S) is coarsely simply connected.*

More generally, if G is a σ–compact locally compact group with d an adapted pseudo-metric, then G is compactly presented if and only if the pseudo-metric space (G, d) is coarsely simply connected.

Proof. The first statement hold thanks to Theorem 4.28.

For the second statement, fix G a σ–compact locally compact group with d an adapted pseudo-metric on G. Suppose first that G is compactly presented, say $G = \langle S \mid R \rangle$ with S compact. Then (G, d_S) is coarsely simply connected, and by Corollary 2.32, there is a metric coarse equivalence between (G, d_S) and (G, d). Coarse simple connectedness being preserved by metric coarse equivalences (Theorem 4.8), we deduce that (G, d) is coarsely simply connected as well, as claimed.

Conversely, assume that (G, d) is coarsely simply connected. In particular, (G, d) is coarsely connected, so that G is compactly generated by Theorem 2.37. Let S be a compact generating set for G. Since (G, d_S) and (G, d) are coarsely equivalent, it follows that (G, d_S) is also coarsely simply connected, whence G is in fact compactly presented. □

We can therefore conclude that compact presentation provides an additional coarse geometric invariant.

Corollary 4.30. *Among σ–compact locally compact groups, being compactly presented is invariant under metric coarse equivalence.*

In particular, among compactly generated locally compact groups, being compactly presented is invariant under quasi-isometry.

Example 4.31. For instance any virtually free group, such as $\mathrm{SL}_2(\mathbb{Z})$, $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, $\mathrm{PSL}_2(\mathbb{Z})$, or D_∞, is finitely presented.

This allows us to deduce how compact presentation behaves when passing to cocompact closed subgroups.

Corollary 4.32. *Let G be a locally compact group, and H a cocompact closed subgroup.*

Then G is compactly presented if and only if H is compactly presented.

Proof. Let d be an adapted pseudo-metric on G. Then the inclusion $(H, d) \hookrightarrow (G, d)$ is a coarse embedding by Corollary 2.32, and it is essentially surjective since H is cocompact. Thus there is a metric coarse equivalence $(H, d) \longrightarrow (G, d)$, so the conclusion follows from Corollary 4.30. □

This corollary has the following consequence.

Corollary 4.33. *Compact groups are compactly presented.*

In fact, our previous results on bounded presentations also show the following stability properties.

Proposition 4.34. *Let*

$$1 \longrightarrow N \xrightarrow{\iota} H \xrightarrow{\pi} Q \longrightarrow 1$$

be a short exact sequence of locally compact groups and continuous homomorphisms.

(i) *If G is compactly presented and N is compactly generated as a normal subgroup of G, then Q is compactly presented.*

(ii) *If G is compactly generated and Q is compactly presented, then N is compactly generated as a normal subgroup of G.*

(iii) *If N and Q are compactly presented, then G is compactly presented.*

Proof. This follows from Lemma 4.25 and Proposition 4.27. □

In the discrete setting, it has for instance the following nice application.

Theorem 4.35. *Finitely generated polycyclic groups are finitely presented.*

Proof. Let thus G be a finitely generated polycyclic group, with a sequence of subgroups

$$H_0 = \{e_G\} \leqslant H_1 \leqslant \cdots \leqslant H_{s-1} \leqslant H_s = G$$

so that $H_i \lhd H_{i+1}$ and the quotient group H_{i+1}/H_i is cyclic, for any $i = 0, \ldots, s-1$. We prove that G is finitely presented by induction on s.

If $s = 0$, there is nothing to show. If $s = 1$ then G is cyclic, thus finitely presented. Suppose then that the result holds for any finitely generated polycyclic group with a sequence of subgroups with length at most $s-1$. Observe then that G fits into a short exact sequence

$$1 \longrightarrow H_{s-1} \longrightarrow G \longrightarrow G/H_{s-1} \longrightarrow 1$$

where H_{s-1} is polycyclic with a sequence of subgroups as in Definition 1.44 whose length does not exceed $s-1$, thus is finitely presented by the inductive hypothesis. On the other hand, the quotient $G/H_{s-1} = H_s/H_{s-1}$ is cyclic, thus finitely presented as well. We conclude that G is finitely presented by Proposition 4.34(iii). □

We conclude with the natural analog of Corollary 3.4 for compact presentation. To state it, we just need one additional observation.

Proposition 4.36. *Let G be a compactly generated locally compact group, S a compact generating set, and d an adapted pseudo-metric on G. Let $c \geq 1$.*

(i) *The inclusion of (G, d) into $\mathrm{Rips}_c^2(G, d_S)$ is a metric coarse equivalence.*

(ii) *If d is moreover geodesically adapted, the inclusion of (G,d) into $\mathrm{Rips}^2_c(G,d_S)$ is a quasi-isometry.*

Proof. (i) We know from Corollary 2.32 that the identity map $(G,d) \longrightarrow (G,d_S)$ is a metric coarse equivalence, and from Proposition 4.19 that $(G,d_S) \hookrightarrow \mathrm{Rips}^2_c(G,d_S)$ is a quasi-isometry. Thus $(G,d) \hookrightarrow \mathrm{Rips}^2_c(G,d_S)$ is a metric coarse equivalence.

(ii) If moreover d is geodesically adapted, the same argument as in (i) replacing Corollary 2.32 by Corollary 2.39 shows that $(G,d) \hookrightarrow \mathrm{Rips}^2_c(G,d_S)$ is a quasi-isometry. ◻

Combining Theorem 4.28, Theorem 4.29 and Proposition 4.36, we deduce the next equivalences.

Corollary 4.37. *Let G be a compactly generated locally compact group, S a compact generating set, and d a geodesically adapted pseudo-metric on G. The following are equivalent.*

(i) *The locally compact group G is compactly presented.*

(ii) *The pseudo-metric space (G,d) is coarsely simply connected.*

(iii) *The inclusion map $(G,d) \hookrightarrow \mathrm{Rips}^2_c(G,d_S)$ is a metric coarse equivalence for all $c \geq 1$.*

(iv) *The inclusion map $(G,d) \hookrightarrow \mathrm{Rips}^2_c(G,d_S)$ is a quasi-isometry for all $c \geq 1$.*

(v) $\mathrm{Rips}^2_c(G,d_S)$ *is simply connected for all c large enough.*

Thus, for compact presentation, Milnor-Schwarz lemma takes the following form.

Theorem 4.38. *Let G be a locally compact group acting geometrically on a pseudo-metric space X.*

Then G is compactly presented if and only if X is coarsely simply connected.

We can therefore conclude that geometric actions on geodesic simply connected metric spaces caracterise compactly presented groups.

Corollary 4.39. *Let G be a locally compact group. The following claims are equivalent.*

(i) *The group G is compactly presented.*

(ii) *There exists a geometric action of G on a non-empty coarsely simply connected pseudo-metric space.*

(iii) *There exists a geometric action of G on a non-empty geodesic simply connected metric space.*

(iv) *There exists a geometric faithful action of G on a non-empty geodesic simply connected metric space.*

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