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Geometric approaches to amenability and unitarisability

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Introduction

As outlined for instance in [2, 4, 13, 24], amenability for groups has grown into a central theme of research in geometric group theory since the 30's, thanks to John Von Neumann and his work on the famous *Banach-Tarski-Hausdorff paradox*. Amenability admits a wide variety of equivalent characterizations, establishing deep connections between group theory and apparently unrelated mathematical areas, such as geometry, functional analysis, probability theory, or ergodic theory.

A particularly interesting link with the notion of unitarisability for groups has attracted much attention since the essential works of Jacques Dixmier in the early 50's. In [11], Dixmier showed that any amenable group is unitarisable, and asked whether the converse holds:

Dixmier's problem. Is every unitarisable group amenable?

The answer to this question is still unknown in full generality, but a lot of progress has been made towards the understanding and the main properties of the class of unitarisable groups. In 1955, Ehrenpreis and Mautner obtained in [14] the first example of a non-unitarisable group, $\mathrm{SL}_2(\mathbb{R})$. Later on, it was showed that non-abelian free groups are also non-unitarisable. It is also known that the class of unitarisable groups enjoys several stability properties, when taking subgroups, quotients, and extensions by amenable groups. All these results are in favour of a positive answer to Dixmier's problem.

Gilles Pisier has been a great contributor to those questions, with in particular two results of major interest, exposed in [25, 26]. The first one is a quantitive measurement of unitarisability, and shows that we can always have a good control on the size of unitarisers for uniformly bounded representations of a unitarisable group. The second one improves this control if the group is amenable.

In 2014, Peter Schlicht obtained in [29, 30] proofs of Pisier's results in a completely different manner, much more geometric, by looking at actions of groups on the cone of positive invertible operators on a Hilbert space. The goal of this thesis is to present these proofs and their consequences.

Along the way, a large number of preliminaries will be required, and we now describe the content of each chapter, with an emphasize on the main results.

Chapter 1 focuses on the theory of bounded linear operators on Hilbert spaces. After recalling general background material concerning Banach and Hilbert spaces, we introduce linear operators, and prove an invertibility criterion (Proposition 1.2). We recall the construction of adjoint operators with the Riesz representation theorem. Afterwards, we study several classes of operators, especially normal, self-adjoint, unitary, positive and isometric operators (Proposition 1.9). Then, we turn to the spectral theory of bounded operators, and introduce the spectrum, resolvent set and spectral radius of an operator. We show that its spectral radius is always bounded by its norm (Proposition 1.12), and that its spectrum is always a compact subset of the complex plane (Proposition 1.14). In fact, it is a compact subset of the real line for self-adjoint operators, and of the positive real numbers for positive operators (Proposition 1.19). We derive several useful properties of the spectral radius (Corollary 1.25) through the *Gelfand's formula* (Theorem 1.24). To pursue, we focus on the functional calculus for bounded self-adjoint operators, and establish its main properties (Theorem 1.30, Theorem 1.35). This allows us to completely characterize self-adjoint, unitary and positive operators through their spectrum (Corollary 1.32). An important application of functional calculus for the rest of the text is the construction of the exponential map, that we present in details (Corollary 1.38, Remark 1.39). We define square roots and polar decompositions, and show their existence and uniqueness for suitable operators (Theorem 1.42). We derive from this a complete description of positive operators on a Hilbert space (Corollary 1.43). We conclude this chapter by looking at another topology on the space of bounded operators on a Hilbert space. This topology is smaller than the norm topology (Lemma 1.48), but still remains Hausdorff (Lemma 1.50). The interest of considering it is to obtain more compact subsets (Theorem 1.52).

Chapter 2 is devoted to the cone of positive invertible operators on a Hilbert space. We explain the terminology "cone" (Lemma 2.1), and we define a natural action of invertible operators on this set, which is transitive and continuous (Lemma 2.3). We endow this set with a metric structure, and we show that invertible operators act by isometries with respect to that metric (Proposition 2.6). Next, we show this metric space is geodesic, and that geodesics are preserved by the action of invertible operators (Lemma 2.8). The second part of this chapter aims at proving a convexity inequality for the distance between those geodesics. In that view, we introduce an order relation on the set of self-adjoint operators (Definition 2.10) and establish several rules of computations for this relation (Proposition 2.11). We prove the *Löwner-Heinz inequality* for positive operators (Theorem 2.14), from which we derive additional operator inequalities, especially the *Jensen's inequality* (Theorem 2.17) for contractions, and a more general form for general operators (Corollary 2.18). We also derive the *Corach-Porta-Recht inequality* (Theorem 2.20) and the *Cordes inequality* (Corollary 2.21), the crucial ingredient for the proof of the convexity inequality (Theorem 2.23).

Chapter 3 is the central part of the thesis. We define representations of groups on Hilbert spaces and several related terminologies. We introduce unitarisability for representations, and we prove it is equivalent to the existence of an invariant inner product on the Hilbert space inducing the same topology as the initial one (Lemma 3.8). We define the class of unitarisable groups. We show it contains finite groups (Corollary 3.9), and that it is closed under taking quotients and subgroups (Proposition 3.10, Proposition 3.12). In particular, induction of representations is presented in details. Next, we prove that amenable groups are unitarisable, recovering the result of Dixmier (Theorem 3.13). On the other hand, we establish also the non-unitarisability of the non-abelian free group on countably many generators (Theorem 3.18), appealing to the concept of derivations. Coupled with the *Ping-Pong lemma* (Lemma 3.19), we deduce that any non-abelian free group is not unitarisable, as well as many linear groups, for instance $SL_2(\mathbb{R})$ (Corollary 3.22). In the following section, we explain how a group representation gives rise to an action of the group on the cone of positive invertible operators, and this action has fixed points if and only if the representation is unitarisable (Lemma 3.23). Exploiting the weak operator topology, we prove that a unitarisable representation can always be unitarised at minimal "cost" (Proposition 3.25). Given a unitarisable representation π of a group G, we introduce a family $(\pi_t)_{t\in[0,1]}$ of representations of G for which we can recover sizes and unitarisers in terms of the corresponding ones for π (Lemma 3.27, Corollary 3.28). In the next subsection, we define the *diameter* of a representation, and we provide an explicit formula for its computation (Proposition 3.31). We can then relate the size of π_t, $t \in [0,1]$, to the size of π (Proposition 3.32). With these results, we show that if a countable family of uniformly bounded representations of a unitarisable group is also uniformly bounded, each representation can be unitarised at low cost, and this cost is uniform throughout the whole family (Proposition 3.35). We then establish Pisier's theorem (Theorem 3.36), and using functional calculus, we relate size of unitarisers for a unitarisable representation to the distance between the identity and the set of fixed points for the induced action (Proposition 3.39). This leads to a geometric translation of Pisier's result (Corollary 3.40), which in turn leads to a geometric formulation of amenability (Corollary 3.41).

Two appendices are added. Appendix A contains basic material about general topology, with which most of the readers will be familiar. It is provided for the sake of completeness, for recalling basic definitions and properties of topological spaces, and for justifying easily many parts in several proofs. Appendix B is an introduction to the theme of amenability for groups. We recall several equivalent formulations of amenability. We explore the class of amenable groups, in order to illustrate clearly similarities with unitarisability in Chapter 3.

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Notations

$\emptyset$	The empty set
$\mathbb{N} = \{0, 1, 2, \dots\}$	The set of natural integers
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\lvert z\rvert$	The modulus of $z \in \mathbb{C}$
$\overline{z}$	The complex conjugate of $z \in \mathbb{C}$
$\mathbb{S}^1 = \{\lambda \in \mathbb{C} : \lvert\lambda\rvert = 1\}$	The unit circle in the complex plane
$K[X]$	The set of polynomials with coefficients in a field K
$A \subset B$	A is a subset of B
$A \cup B$	The union of two sets A and B
$A \sqcup B$	The disjoint union of two sets A and B
$A \cap B$	The intersection of two sets A and B
$A \Delta B$	The symmetric difference of A and B, given by $(A \cup B) \setminus (A \cap B)$
$\mathbf{1}_X$	The indicator function of the set X
Id_X	The identity map on a set X
$\lvert X\rvert$	The cardinality of the set X
$\mathcal{P}_{\mathrm{s}}(X)$	The set of subsets of X
$\mathrm{Vect}(X)$	The vector subspace generated by X
$g \circ f$	The composition of two functions f and g
f^{-1}	The inverse of a bijective map f
$\lvert f\rvert$	The modulus of a $\mathbb{C}-$valued map f
$\mathcal{F}(X, Y)$	The set of maps from X to Y
$\exp\colon \mathbb{R} \longrightarrow (0, \infty)$	The exponential map from $\mathbb{R}$ to $(0, \infty)$
$\ln\colon (0, \infty) \longrightarrow \mathbb{R}$	The inverse of the exponential map
a.e.	almost everywhere
e_G	The neutral element of a group G
$H \leqslant G$	H is a subgroup of G
$N \lhd G$	N is a normal subgroup of G
G/H	The quotient set of G by a subgroup H
G/N	The quotient group of G by a normal subgroup N

$[G : H]$	The index of H in G, defined as the cardinality of G/H
δ_g	The Dirac mass at $g \in G$, equals to $\mathbf{1}_{\{g\}} : G \longrightarrow \mathbb{C}$
$\mathrm{GL}_n(K)$	The group of invertible $n \times n$ matrices with entries in a field K
$\mathrm{GL}_n(\mathbb{Z})$	The group of invertible $n \times n$ matrices with integer entries
$\mathrm{SL}_n(K)$	The group of determinant one $n \times n$ matrices with entries in a field K
$\mathrm{SL}_n(\mathbb{Z})$	The group of determinant one $n \times n$ matrices with integer entries
$\cong$	An isomorphism between two objects in the same category

Contents

1. Bounded operators on Hilbert spaces

In this first chapter, we develop the general theory about bounded linear operators on a complex Hilbert space. After having studied several types of operators, we define the spectrum and the spectral radius of a bounded operator and establish their main properties. We introduce the functional calculus, and we show the existence of square roots for bounded positive operators and polar decompositions for invertible operators. To conclude, we define and study a new topology on the space of bounded linear operators on a Hilbert space.

Before focusing on Hilbert spaces, we recall terminologies and results from the more general setting of normed and Banach spaces. We will not show all the results below, but we indicate references for the proofs.

Recall first that a *norm* on a $\mathbb{C}$−vector space X is a map $\|\cdot\|_X : X \longrightarrow [0,\infty)$ so that

(i) $\|x\|_X = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\|_X = |\lambda|\|x\|_X$ for all $x \in X, \alpha \in \mathbb{C}$.

(iii) $\|x+y\|_X \leq \|x\|_X + \|y\|_X$ for all $x, y \in X$.

Point (iii) is called the *triangle inequality*, and the pair $(X, \|\cdot\|_X)$ is called a *normed space*.

Note that if $(X, \|\cdot\|_X)$ is a normed space, then one has also a second triangle inequality, namely

$$\big|\|x\|_X - \|y\|_X\big| \leq \|x-y\|_X$$

for all $x, y \in X$. Indeed if $x, y \in X$, then $\|x\|_X = \|(x-y)+y\|_X \leq \|x-y\|_X + \|y\|_X$ by (iii), so $\|x\|_X - \|y\|_X \leq \|x-y\|_X$. Switching the roles of x and y leads to $\|y\|_X - \|x\|_X \leq \|x-y\|_X$, whence $\big|\|x\|_X - \|y\|_X\big| \leq \|x-y\|_X$ as announced.

A normed space $(X, \|\cdot\|_X)$ is a metric space for the metric $\mathrm{d}_X : X \times X \longrightarrow [0,\infty)$ defined by

$$\mathrm{d}_X(x,y) = \|x-y\|_X,\ x, y \in X$$

and thus a topological space for the topology induced by d_X (see Appendix A), that we denote $\tau_{\|\cdot\|_X}$. For $x \in X$ and $r > 0$, we write $B_{\|\cdot\|_X}(x,r)$ (resp. $B'_{\|\cdot\|_X}(x,r)$) for the open (resp. closed) ball of radius $r > 0$ around $x \in X$. Moreover, the norm is a continuous map for this topology.

Proof. Let $\varepsilon > 0$, and note that

$$\|\cdot\|_X^{-1}([0,\varepsilon)) = \{x \in X : \|x\|_X < \varepsilon\} = \{x \in X : \mathrm{d}_X(x,0) < \varepsilon\} = B_{\|\cdot\|_X}(0,\varepsilon)$$

is an open set in X, by definition of $\tau_{\|\cdot\|_X}$. Thus $\|\cdot\|_X : X \longrightarrow [0,\infty)$ is continuous. □

A sequence $(x_n)_{n\in\mathbb{N}}$ in a normed space $(X, \|\cdot\|_X)$ is called a *Cauchy sequence* with respect to d_X if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that

$$n, m \geq N \Longrightarrow \mathrm{d}_X(x_n, x_m) < \varepsilon.$$

We say that $(X, \|\cdot\|_X)$ is a *Banach space* if the metric space (X, d_X) is complete, *i.e.* if any Cauchy sequence with respect to d_X has a limit in X.

Most important examples of Banach spaces are as follows.

Examples. (i) If X is a finite-dimensional vector space, any two norms $\|\cdot\|_X, \|\cdot\|'_X$ on X are *equivalent*, in the sense that there are constants $c, c' > 0$ so that

$$c\|x\|_X \leq \|x\|'_X \leq c'\|x\|_X$$

for all $x \in X$ (see *e.g.* [3, theorem 1.2.5], [6, theorem 3.3.1]). Moreover, X is complete for any choice of norm [3, corollary 1.2.6].

(ii) For $a, b \in \mathbb{R}$, $a < b$, the space $C([a, b])$ of continuous functions on the interval $[a, b]$, equipped with the norm

$$\|f\|_\infty := \sup_{t\in[a,b]} |f(t)|$$

is a Banach space [15, section 1.5]. More generally, if X is a compact Hausdorff space, the space $C(X)$ of continuous functions on X with the supremum norm is a Banach space [6, example 3.1.6].

(iii) Consider a measure space $(X, \mathbb{A}, \mu)$ and, for $1 \leq p < \infty$, the vector space

$$\mathcal{L}^p(X, \mathbb{A}, \mu) := \left\{f : X \longrightarrow \mathbb{C} \text{ measurable} \;\middle|\; \int_X |f|^p \,\mathrm{d}\mu < \infty\right\}$$

and also

$$\mathcal{L}^\infty(X, \mathbb{A}, \mu) := \{f : X \longrightarrow \mathbb{C} \text{ measurable} \mid \exists C \geq 0,\ |f| \leq C\ \mu-\text{a.e.}\}.$$

For every $p \in [1, \infty]$, we define an equivalence relation $\sim$ on $\mathcal{L}^p(X, \mathbb{A}, \mu)$ by $f \sim g$ if and only if $f = g$ μ–a.e., and we form the quotient

$$L^p(X, \mathbb{A}, \mu) := \mathcal{L}^p(X, \mathbb{A}, \mu)/\!\sim.$$

Identifying an equivalence class with one of its representatives, we define a norm $\|\cdot\|_p$ on the quotient via

$$\|f\|_p := \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p}$$

for every $1 \leq p < \infty$ and $f \in L^p(X, \mathbb{A}, \mu)$, as well as

$$\|f\|_\infty := \inf\{C \in [0, \infty] : |f| \leq C\ \mu-\text{a.e.}\}$$

if $f \in L^\infty(X, \mathbb{A}, \mu)$. It can be shown that, for every $p \in [1, \infty]$, $(L^p(X, \mathbb{A}, \mu), \|\cdot\|_p)$ is a complete normed space [20, theorem 4.2.2]. If X is countable, we denote this space

$\ell^p(X, \mathbb{A}, \mu)$, or even $\ell^p(X)$ if the underlying $\sigma-$algebra and measure are clear from the context.

If $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are two normed spaces, a *linear operator* between X and Y is a map $A\colon X \longrightarrow Y$ so that

$$A(\lambda x + y) = \lambda A(x) + A(y)$$

for all $x \in X$, $y \in Y$ and $\lambda \in \mathbb{C}$.

In the sequel, we merely write Ax for the image of $x \in X$ under A.

Such an operator is continuous if and only if it is *bounded*, *i.e.* there exists a constant $C > 0$ so that $\|Ax\|_Y \leq C\|x\|_X$ for any $x \in X$ ([3, theorem 1.2.2], [6, proposition 2.2.1]). The set of bounded linear operators between two normed spaces X and Y will be denoted $\mathcal{B}(X,Y)$, or simply $\mathcal{B}(X)$ if $X = Y$. It has the structure of a vector space over $\mathbb{C}$ for the sum and scalar multiplication defined as

$$(A+B)(x) := Ax + Bx,\ \ (\lambda A)(x) := \lambda Ax$$

for any $A, B \in \mathcal{B}(X,Y)$, $x \in X$ and $\lambda \in \mathbb{C}$. The map $\|\cdot\|_{\text{op}}\colon \mathcal{B}(X,Y) \longrightarrow [0,\infty)$, defined as

$$\|A\|_{\text{op}} := \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\|_X\leq 1} \|Ax\|_Y = \sup_{\|x\|_X=1} \|Ax\|_Y$$

for any $A \in \mathcal{B}(X,Y)$, is a norm on $\mathcal{B}(X,Y)$ [6, proposition 2.1.2]. Moreover, for $A \in \mathcal{B}(X,Y)$ one has

$$\|Ax\|_Y \leq \|A\|_{\text{op}}\|x\|_X$$

for all $x \in X$, and $\|\cdot\|_{\text{op}}$ is *submultiplicative*, *i.e.* if X, Y, and Z are three normed spaces and $A \in \mathcal{B}(X,Y)$, $B \in \mathcal{B}(Y,Z)$, then $B \circ A \in \mathcal{B}(X,Z)$ and

$$\|B \circ A\|_{\text{op}} \leq \|A\|_{\text{op}}\|B\|_{\text{op}}.$$

Proof. The inequality $\|Ax\|_Y \leq \|A\|_{\text{op}}\|x\|_X$ is a consequence of the definition of $\|A\|_{\text{op}}$ for $x \neq 0$, and it clearly holds if $x = 0$. Now, if X, Y and Z are normed spaces, $A \in \mathcal{B}(X,Y)$, $B \in \mathcal{B}(Y,Z)$ and $x \in X$ with $\|x\|_X \leq 1$, one has

$$\|(B \circ A)x\|_Z \leq \|B\|_{\text{op}}\|Ax\|_Y \leq \|A\|_{\text{op}}\|B\|_{\text{op}}\|x\|_X \leq \|A\|_{\text{op}}\|B\|_{\text{op}}$$

so that $\|B \circ A\|_{\text{op}} \leq \|A\|_{\text{op}}\|B\|_{\text{op}}$ as announced. □

When there is no risk of confusion we simply write $\|x\|$ for the norm of an element $x \in X$ of a normed space, or $\|A\|$ for the operator norm $\|A\|_{\text{op}}$ of a bounded linear operator between two normed spaces. We also write BA for the composition of the two operators A and B.

The next theorem guarantees that $\mathcal{B}(X,Y)$ is in fact a complete space, provided Y is complete.

Theorem. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces, and assume that Y is a Banach space. Then $\mathcal{B}(X,Y)$ is a Banach space.

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(X,Y)$. Let $\varepsilon > 0$ and $x \in X$. Using the assumption, we find $N \in \mathbb{N}$ so that $\|A_n - A_m\| < \varepsilon$ for all $n, m \geq N$, and this implies

$$\|A_n x - A_m x\|_Y \leq \|A_n - A_m\| \|x\|_X < \varepsilon \|x\|_X$$

for all $n, m \geq N$. Thus $(A_n x)_{n\in\mathbb{N}}$ is Cauchy in Y, and since the latter is complete, $(A_n x)_{n\in\mathbb{N}}$ has a limit in Y, that we denote Ax. This defines a linear map between X and Y.

It remains to see that A is bounded and that $(A_n)_{n\in\mathbb{N}}$ converges to A in $\mathcal{B}(X,Y)$. Fix $\varepsilon > 0$. As seen above, we find $N \in \mathbb{N}$ so that $\|A_n - A_m\| < \varepsilon$ for all $n, m \geq N$, and in particular

$$\|A_n x - A_m x\|_Y \leq \|A_n - A_m\| \|x\|_X < \varepsilon \|x\|_X$$

for all $n, m \geq N$ and all $x \in X$. Thus

$$\|Ax - A_n x\|_Y = \lim_{m\to\infty} \|A_m x - A_n x\|_Y \leq \varepsilon \|x\|_X$$

for all $n \geq N$ and $x \in X$, by continuity of the norm. In particular we deduce

$$\|Ax\|_Y \leq \|Ax - A_N x\|_Y + \|A_N x\|_Y \leq (\|A_N\| + \varepsilon)\|x\|_X$$

for all $x \in X$, whence A is bounded. Additionally, from the above we have

$$\frac{\|Ax - A_n x\|_Y}{\|x\|_X} \leq \varepsilon$$

for $n \geq N$ and $x \neq 0$, and it follows that

$$\|A - A_n\| = \sup_{x\neq 0} \frac{\|Ax - A_n x\|_Y}{\|x\|_X} \leq \varepsilon$$

for all $n \geq N$. As $\varepsilon > 0$ was arbitrary, this shows that $A_n \to A$ in $\mathcal{B}(X,Y)$, and concludes the proof. □

In particular, if X is a normed space, choosing $Y = \mathbb{C}$ in the previous theorem shows that $\mathcal{B}(X,\mathbb{C})$ is a Banach space. This space is called the *dual space* of X, and is often denoted X^*. Its elements are called *linear functionals on* X.

If $\mathcal{H}$ is a complex vector space, a *hermitian inner product* on $\mathcal{H}$ is a map

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} \colon \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$$

so that

(i) $\langle \lambda u + \mu v, w\rangle_{\mathcal{H}} = \lambda\langle u, w\rangle_{\mathcal{H}} + \mu\langle v, w\rangle_{\mathcal{H}}$ for all $u, v, w \in \mathcal{H}$, $\lambda, \mu \in \mathbb{C}$.

(ii) $\langle v, u\rangle_{\mathcal{H}} = \overline{\langle u, v\rangle_{\mathcal{H}}}$ for all $u, v \in \mathcal{H}$.

(iii) $\langle u, u\rangle_{\mathcal{H}} \geq 0$ for all $u \in \mathcal{H}$, and $\langle u, u\rangle_{\mathcal{H}} = 0$ implies $u = 0$.

The pair $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is called a *pre-Hilbert space*. It follows from points (i) and (ii) that a hermitian inner product is *anti-linear* in the second variable, *i.e.*

$$\langle u, \lambda v + \mu w\rangle_{\mathcal{H}} = \overline{\lambda}\langle u, v\rangle_{\mathcal{H}} + \overline{\mu}\langle u, w\rangle_{\mathcal{H}}$$

for all $u, v, w \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$. It also follows from (ii) that $\langle u, u\rangle_{\mathcal{H}}$ is a real number for any $u \in \mathcal{H}$, and it makes sense to talk about its sign in (iii). Additionally, (i) implies that $\langle u, 0\rangle_{\mathcal{H}} = \langle 0, u\rangle_{\mathcal{H}} = 0$ for any $u \in \mathcal{H}$, and thus $\langle u, u\rangle_{\mathcal{H}} = 0$ if and only if $u = 0$.

As explained in [6, 12, 13, 15], a pre-Hilbert space can be turned into a normed space, by defining the map

$$\begin{aligned} \|\cdot\|_{\mathcal{H}} \colon \mathcal{H} &\longrightarrow [0,\infty) \\ u &\longmapsto \sqrt{\langle u, u\rangle_{\mathcal{H}}}. \end{aligned}$$

We call $\mathcal{H}$ a *Hilbert space* if the normed space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is complete. Phrased differently, a Hilbert space is a Banach space for which the norm is *induced by* (or *derived from*) a hermitian inner product.

Examples. (i) Let $n \geq 1$. The usual norm on $\mathbb{C}^n$, defined as

$$\|x\| := \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, derives from the inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$ defined as

$$\langle x, y\rangle_{\mathbb{C}^n} := \sum_{i=1}^{n} x_i \overline{y_i}$$

for any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n$. Coupled with the fact that $\mathbb{C}^n$ is complete, this tells us that $\mathbb{C}^n$ is a Hilbert space.

(ii) If $(X, \mathbb{A}, \mu)$ is a measure space, the norm $\|\cdot\|_2$ on $L^2(X, \mathbb{A}, \mu)$ defined above is easily seen to derive from the inner product $\langle\cdot,\cdot\rangle_2$ given by

$$\langle f, g\rangle_2 := \int_X f(x)\overline{g(x)}\, \mathrm{d}\mu(x)$$

for any $f, g \in L^2(X, \mathbb{A}, \mu)$. Thus $L^2(X, \mathbb{A}, \mu)$ is a Hilbert space.

A consequence of these definitions is the so-called *Cauchy-Schwarz inequality* ([3, lemma 1.4.2], [13, lemma 1.3], [15, theorem 2.1.1]), which ensures that

$$|\langle u, v\rangle_{\mathcal{H}}| \leq \|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$$

for any $u, v \in \mathcal{H}$, as well as the continuity of the inner product in each variable, *i.e.* for any $v \in \mathcal{H}$ the maps $u \longmapsto \langle u, v\rangle_{\mathcal{H}}$, $u \longmapsto \langle v, u\rangle_{\mathcal{H}}$ are continuous from $\mathcal{H}$ to $\mathbb{C}$.

Proof. Let $v \in \mathcal{H}$. We show the continuity of $u \longmapsto \langle u, v\rangle$, and the other one is very similar. Let then $u_0 \in \mathcal{H}$ and $\varepsilon > 0$. Set $\delta := \frac{\varepsilon}{1+\|v\|} > 0$, and observe that if $\|u - u_0\| < \delta$, then

$$|\langle u, v\rangle - \langle u_0, v\rangle| = |\langle u - u_0, v\rangle| \leq \|u - u_0\|\|v\| < \delta\ \|v\| < \varepsilon$$

by the Cauchy-Schwarz inequality. Hence $u \longmapsto \langle u, v\rangle$ is continuous at any $u_0 \in \mathcal{H}$, and therefore is continuous on $\mathcal{H}$. □

A Hilbert space $\mathcal{H}$ is called *separable* if it contains a countable dense subset. In this thesis, unless stated otherwise, any involved Hilbert space is assumed to be separable.

If $\mathcal{H}_1, \mathcal{H}_2$ are two Hilbert spaces, there is a natural Hilbert space structure on the cartesian product $\mathcal{H}_1 \times \mathcal{H}_2$ [21, section 2.6], with the inner product defined as

$$\langle (u_1, u_2), (v_1, v_2) \rangle_{\mathcal{H}_1 \times \mathcal{H}_2} := \langle u_1, v_1 \rangle_{\mathcal{H}_1} + \langle u_2, v_2 \rangle_{\mathcal{H}_2}$$

for any $u_1, v_1 \in \mathcal{H}_1$, $u_2, v_2 \in \mathcal{H}_2$. We call $(\mathcal{H}_1 \times \mathcal{H}_2, \langle \cdot, \cdot \rangle_{\mathcal{H}_1 \times \mathcal{H}_2})$ the *direct sum* of $\mathcal{H}_1$ and $\mathcal{H}_2$, and we denote this space $\mathcal{H}_1 \oplus \mathcal{H}_2$. The norm induced by the above inner product is

$$\|(u_1, u_2)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2} := \sqrt{\|u_1\|^2_{\mathcal{H}_1} + \|u_2\|^2_{\mathcal{H}_2}}, \ u_1 \in \mathcal{H}_1, \ u_2 \in \mathcal{H}_2.$$

It is useful to observe that for any $u_1 \in \mathcal{H}_1, u_2 \in \mathcal{H}_2$, we have the inequalities

$$\frac{1}{\sqrt{2}}(\|u_1\|_{\mathcal{H}_1} + \|u_2\|_{\mathcal{H}_2}) \leq \|(u_1, u_2)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2} \leq \|u_1\|_{\mathcal{H}_1} + \|u_2\|_{\mathcal{H}_2}.$$

The first one is a consequence of the inequality $(a+b)^2 \leq 2(a^2 + b^2)$, $a, b \geq 0$, while the second follows from $a^2 + b^2 \leq (a+b)^2$, also valid for $a, b \geq 0$.

Now, if $A_1 \in \mathcal{B}(\mathcal{H}_1)$, $A_2 \in \mathcal{B}(\mathcal{H}_2)$, we may define a linear operator C on $\mathcal{H}_1 \oplus \mathcal{H}_2$ by the formula

$$C(u_1, u_2) := (Au_1, Bu_2), \ u_1 \in \mathcal{H}_1, u_2 \in \mathcal{H}_2.$$

We claim that $\|C\| = \max(\|A\|, \|B\|)$.

Proof. Let $c := \max(\|A\|, \|B\|)$. For any $u_1 \in \mathcal{H}_1, u_2 \in \mathcal{H}_2$, we have

$$\begin{aligned}
\|C(u_1, u_2)\|^2_{\mathcal{H}_1 \oplus \mathcal{H}_2} &= \|(Au_1, Bu_2)\|^2_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\
&= \|Au_1\|^2_{\mathcal{H}_1} + \|Bu_2\|^2_{\mathcal{H}_2} \\
&\leq \|A\|^2 \|u_1\|^2_{\mathcal{H}_1} + \|B\|^2 \|u_2\|^2_{\mathcal{H}_2} \\
&\leq c^2 \|u_1\|^2_{\mathcal{H}_1} + c^2 \|u_2\|^2_{\mathcal{H}_2} \\
&= c^2 \|(u_1, u_2)\|^2_{\mathcal{H}_1 \oplus \mathcal{H}_2}
\end{aligned}$$

whence $\|C\| \leq c$. On the other hand, by definition of $\|C\|$, we have

$$\|C\| \geq \sup_{u_1 \neq 0} \frac{\|C(u_1, 0)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2}}{\|(u_1, 0)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2}} = \sup_{u_1 \neq 0} \frac{\|(Au_1, 0)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2}}{\|(u_1, 0)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2}} = \sup_{u_1 \neq 0} \frac{\|Au_1\|_{\mathcal{H}_1}}{\|u_1\|_{\mathcal{H}_1}} = \|A\|$$

and similarly $\|C\| \geq \|B\|$. This leads to $\|C\| \geq c$, and thus $\|C\| = c$. □

By induction, these constructions extend to an arbitrary finite number of Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$. With additional work, this can in fact be extended to an infinite collection $\{\mathcal{H}_n\}_{n \in \mathbb{N}}$ of complex Hilbert spaces. More precisely, given such a collection, the *direct sum* $\bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$ is the set

$$\Big\{(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathcal{H}_n \ : \ \sum_{n \in \mathbb{N}} \|x_n\|^2_{\mathcal{H}_n} < \infty\Big\} \subset \prod_{n \in \mathbb{N}} \mathcal{H}_n$$

whose vector space structure over $\mathbb{C}$ is given coordinatewise and equipped with the norm

$$\|(x_n)_{n\in\mathbb{N}}\|_{\bigoplus_{n\in\mathbb{N}}\mathcal{H}_n} := \sqrt{\sum_{n\in\mathbb{N}} \|x_n\|^2_{\mathcal{H}_n}}, \quad (x_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n.$$

This norm derives from the inner product on $\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ given by

$$\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\rangle_{\bigoplus_{n\in\mathbb{N}}\mathcal{H}_n} := \sum_{n\in\mathbb{N}} \langle x_n, y_n\rangle_{\mathcal{H}_n}.$$

for all $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$. The above series is well-defined, in fact even absolutely convergent, thanks to Cauchy-Schwartz inequality, as

$$\sum_{n\in\mathbb{N}} |\langle x_n, y_n\rangle_{\mathcal{H}_n}| \leq \sum_{n\in\mathbb{N}} \|x_n\|_{\mathcal{H}_n}\|y_n\|_{\mathcal{H}_n} \leq \sqrt{\sum_{n\in\mathbb{N}} \|x_n\|^2_{\mathcal{H}_n}}\sqrt{\sum_{n\in\mathbb{N}} \|y_n\|^2_{\mathcal{H}_n}} < \infty$$

for all $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$.

This indeed gives rise to a complete normed space and as above if $T_n \in \mathcal{B}(\mathcal{H}_n)$ for any $n \in \mathbb{N}$, we may define a linear operator T on the direct sum by setting

$$T(x_n)_{n\in\mathbb{N}} := (T_n x_n)_{n\in\mathbb{N}}$$

and as in the case of two Hilbert spaces, we have

$$\|T\| = \sup_{n\in\mathbb{N}} \|T_n\|.$$

We refer to [21, section 2.6] for the proof of these two facts.

In the sequel we will also require several facts about orthonormal bases in Hilbert spaces, for which the proofs can be found in [6, sections 1.4 and 1.5], or in [15, section 2.1].

First of all, a sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ is an *orthonormal system* if

$$\langle e_i, e_j\rangle_{\mathcal{H}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for any $i, j \in \mathbb{N}$. It is furthermore a *complete* orthonomal system if the subspace

$$\mathrm{Vect}((e_n)_{n\in\mathbb{N}}) := \left\{ \sum_{k=0}^{n} \lambda_k e_k : n \in \mathbb{N},\ \lambda_0, \dots, \lambda_n \in \mathbb{C} \right\}$$

is dense in $\mathcal{H}$. The Gram-Schmidt orthogonalization procedure, as described in [15], proves then the following.

Proposition. A Hilbert space $\mathcal{H}$ is separable if and only if it has a complete orthonormal system $(e_n)_{n\in\mathbb{N}}$.

A sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ is called a *basis* if for all $u \in \mathcal{H}$, there exists a sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}}$ so that

$$u = \sum_{n\in\mathbb{N}} \lambda_n e_n := \lim_{n\to\infty} \sum_{k=0}^{n} \lambda_k e_k.$$

A basis is said to be *orthonormal* if it is an orthonormal system. From the above proposition, we deduce the next theorem.

Theorem. Any separable Hilbert space has an orthonormal basis. Moreover, an orthonormal system $(e_n)_{n\in\mathbb{N}}$ is a basis if and only if

$$\|u\|^2 = \sum_{n\in\mathbb{N}} |\langle u, e_n\rangle|^2$$

for all $u \in \mathcal{H}$.

This last identity is usually called the *Parseval's identity*.

We can thus define the *dimension* of a separable Hilbert space $\mathcal{H}$ as the cardinality of an orthonormal basis of $\mathcal{H}$. This is a well-defined quantity, since any two bases of a Hilbert space have the same cardinality [6, proposition 1.4.14]. It follows from the previous results that for infinite dimensional Hilbert spaces, separability is equivalent to have countable dimension [6, theorem 1.4.16].

Finally, if $(\mathcal{H}_1, \langle\cdot,\cdot\rangle_{\mathcal{H}_1}), (\mathcal{H}_2, \langle\cdot,\cdot\rangle_{\mathcal{H}_2})$ are two Hilbert spaces, a map

$$T\colon (\mathcal{H}_1, \langle\cdot,\cdot\rangle_{\mathcal{H}_1}) \longrightarrow (\mathcal{H}_2, \langle\cdot,\cdot\rangle_{\mathcal{H}_2})$$

is a *unitary equivalence* if it is a linear homeomorphism so that

$$\langle Tu, Tv\rangle_{\mathcal{H}_2} = \langle u, v\rangle_{\mathcal{H}_1}$$

for all $u, v \in (\mathcal{H}_1, \langle\cdot,\cdot\rangle_{\mathcal{H}_1})$.

Theorem. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two separable infinite dimensional Hilbert spaces, there exists a unitary equivalence $T\colon \mathcal{H}_1 \longrightarrow \mathcal{H}_2$.

1.1 Linear operators and their adjoints

Fix, for the rest of this chapter, a complex Hilbert space $\mathcal{H}$ with a hermitian inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and the induced norm $\|\cdot\|_{\mathcal{H}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathcal{H}}}$.

When no confusion is possible we will drop the index $\mathcal{H}$ from the inner product or the norm.

Definition 1.1. Let $A \in \mathcal{B}(\mathcal{H})$.
Its kernel and its image (or range) are defined as

$$\mathrm{Ker}(A) := \{u \in \mathcal{H} : Au = 0\}, \ \mathrm{Im}(A) := \{Au : u \in \mathcal{H}\}.$$

Those are vector subspaces of $\mathcal{H}$, and $\mathrm{Ker}(A) \subset \mathcal{H}$ is closed.

Furthermore, an operator A is invertible if there is a bounded operator $B \colon \mathcal{H} \longrightarrow \mathcal{H}$ so that $AB = BA = \mathrm{Id}_{\mathcal{H}}$, or equivalently if $\mathrm{Ker}(A) = \{0\}$ and $\mathrm{Im}(A) = \mathcal{H}$. We will write $\mathrm{Aut}(\mathcal{H})$ for the set of bounded invertible linear operators on $\mathcal{H}$. It is not a vector subspace of $\mathcal{B}(\mathcal{H})$, because the zero operator is not invertible. Nonetheless, note that if $A, B \in \mathrm{Aut}(\mathcal{H})$, then $AB \in \mathrm{Aut}(\mathcal{H})$ with

$$(AB)^{-1} = B^{-1}A^{-1}$$

and also, when it exists, the inverse of a bounded linear operator is bounded(1). Thus $\mathrm{Aut}(\mathcal{H})$ is a group with respect to composition of operators.

Our first proposition is a useful criteria to determine whether an operator is invertible or not.

Proposition 1.2. Let $A \in \mathcal{B}(\mathcal{H})$. Then A is invertible if and only if $\mathrm{Im}(A)$ is dense in $\mathcal{H}$ and there exists $C > 0$ so that $\|Au\| \geq C\|u\|$ for all $u \in \mathcal{H}$.

Proof. To start, suppose that A is invertible. Then $\mathrm{Im}(A) = \mathcal{H}$ is dense in $\mathcal{H}$, and moreover

$$\|u\| = \|A^{-1}Au\| \leq \|A^{-1}\|\|Au\|$$

for any $u \in \mathcal{H}$. Hence the second condition holds with $C := \frac{1}{\|A^{-1}\|} > 0$.

Conversely, observe that the existence of $C > 0$ forces $\mathrm{Ker}(A) = \{0\}$, because if $u \neq 0$, then $\|u\| > 0$ and it follows $\|Au\| \geq C\|u\| > 0$, so $Au \neq 0$. Now if $(Au_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{Im}(A)$ converging to $v \in \mathcal{H}$, it is a Cauchy sequence and, for $n, m \in \mathbb{N}$, the inequality

$$\|u_n - u_m\| \leq C\|A(u_n - u_m)\| = C\|Au_n - Au_m\|$$

shows that $(u_n)_{n\in\mathbb{N}}$ is also Cauchy, and therefore converges to some $u \in \mathcal{H}$ by completeness. The continuity of A provides then

$$v = \lim_{n\to\infty} Au_n = A(\lim_{n\to\infty} u_n) = Au \in \mathrm{Im}(A).$$

We deduce that $\mathrm{Im}(A)$ is closed, *i.e.* $\overline{\mathrm{Im}(A)} = \mathrm{Im}(A)$. As $\mathrm{Im}(A)$ is also dense by assumption, we conclude that $\mathrm{Im}(A) = \overline{\mathrm{Im}(A)} = \mathcal{H}$, and A is invertible. □

The next result will also be of great help.

(1)This is a consequence of the open mapping theorem, a standard result in functional analysis. See for instance [3, theorem 2.2.1], [6, theorem 12.1], or [15, theorem 9.2.1].

Lemma 1.3. Let $A \in \mathcal{B}(\mathcal{H})$. The following are equivalent.

(i) $A = 0$.

(ii) $\langle Au, v\rangle = 0$ for all $u, v \in \mathcal{H}$.

(iii) $\langle Au, u\rangle = 0$ for all $u \in \mathcal{H}$.

Proof. (i) $\Longrightarrow$ (ii) is immediate, and (ii) $\Longrightarrow$ (iii) follows by putting $v = u$ in (ii).

(iii) $\Longrightarrow$ (ii) : Let $u, v \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Applying (iii) with $u + \lambda v$ yields to

$$0 = \langle A(u + \lambda v), u + \lambda v\rangle = \lambda\langle Av, u\rangle + \overline{\lambda}\langle Au, v\rangle.$$

In particular, using $\lambda = 1$ and $\lambda = i$ we have $\langle Av, u\rangle = -\langle Au, v\rangle$ and $\langle Av, u\rangle = \langle Au, v\rangle$ for all $u, v \in \mathcal{H}$. This proves $\langle Au, v\rangle = 0$ for all $u, v \in \mathcal{H}$, and (ii) holds.

(ii) $\Longrightarrow$ (i) : Fix $u \in \mathcal{H}$. Using (ii) with $v = Au$ gives $\|Au\|^2 = \langle Au, Au\rangle = 0$, so $Au = 0$. As $u \in \mathcal{H}$ was arbitrary, $A = 0$. ◻

For a given bounded operator $A\colon \mathcal{H} \longrightarrow \mathcal{H}$, let $v \in \mathcal{H}$, and consider the linear functional

$$\varphi_v\colon \mathcal{H} \longrightarrow \mathbb{C},\ \ \varphi_v(u) := \langle Au, v\rangle.$$

The Cauchy-Schwarz inequality implies that φ_v is bounded, and the Riesz representation theorem ([3, theorem 1.4.4], [6, theorem 1.3.4], [13, Theorem A.3]) guarantees the existence and the uniqueness of an element $A^*v \in \mathcal{H}$, depending on v, so that

$$\langle Au, v\rangle = \varphi_v(u) = \langle u, A^*v\rangle$$

for all $u \in \mathcal{H}$ and $\|\varphi_v\| = \|A^*v\|$.

This defines a map $A^*\colon \mathcal{H} \longrightarrow \mathcal{H}$ which is linear, as for any $v_1, v_2 \in \mathcal{H}$ and any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\langle Au, v_1 + \lambda v_2\rangle &= \langle Au, v_1\rangle + \overline{\lambda}\langle Au, v_2\rangle \\ &= \langle u, A^*v_1\rangle + \overline{\lambda}\langle u, A^*v_2\rangle \\ &= \langle u, A^*v_1 + \lambda A^*v_2\rangle\end{aligned}$$

for all $u \in \mathcal{H}$. Thus, by uniqueness, $A^*(v_1 + \lambda v_2) = A^*v_1 + \lambda A^*v_2$, and A^* is linear. Moreover, A^* is bounded as

$$\begin{aligned}\|A^*v\| &= \|\varphi_v\| \\ &= \sup_{\|u\|\le 1} |\langle Au, v\rangle| \\ &\le \sup_{\|u\|\le 1} \|A\|\|u\|\|v\| \\ &\le \|A\|\|v\|\end{aligned}$$

for all $v \in \mathcal{H}$, by the Cauchy-Schwarz inequality. Hence $\|A^*\| \le \|A\|$.

We have thus proved the result below.

Theorem 1.4. Let $A \in \mathcal{B}(\mathcal{H})$.
There exists a unique bounded linear operator $A^* : \mathcal{H} \longrightarrow \mathcal{H}$ so that

$$\langle Au, v\rangle = \langle u, A^*v\rangle$$

for all $u, v \in \mathcal{H}$. Moreover, $\|A^*\| \leq \|A\|$.

The operator A^* is called the *adjoint* of A. Here are several basic properties for computing adjoints.

Proposition 1.5. (i) $\mathrm{Id}_{\mathcal{H}}^* = \mathrm{Id}_{\mathcal{H}}$, and $(A^*)^* = A$ for all $A \in \mathcal{B}(\mathcal{H})$.

(ii) $(A + \lambda B)^* = A^* + \overline{\lambda} B^*$ for all $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$.

(iii) $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{B}(\mathcal{H})$.

(iv) $\|A^*\| = \|A\|$, and $\|A^*A\| = \|AA^*\| = \|A\|^2$ for all $A \in \mathcal{B}(\mathcal{H})$.

Proof. (i) For any $u, v \in \mathcal{H}$, we have $\langle u, \mathrm{Id}_{\mathcal{H}}(v)\rangle = \langle u, v\rangle = \langle \mathrm{Id}_{\mathcal{H}}(u), v\rangle$, so necessarily $\mathrm{Id}_{\mathcal{H}}^* = \mathrm{Id}_{\mathcal{H}}$. In the same way, if $A \in \mathcal{B}(\mathcal{H})$, we compute that

$$\langle u, Av\rangle = \overline{\langle Av, u\rangle} = \overline{\langle v, A^*u\rangle} = \langle A^*u, v\rangle$$

for all $u, v \in \mathcal{H}$, which implies $A = (A^*)^*$.
(ii) Fix $u, v \in \mathcal{H}$, and observe that

$$\langle u, (A^* + \overline{\lambda} B^*)v\rangle = \langle u, A^*v\rangle + \lambda\langle u, B^*v\rangle = \langle Au, v\rangle + \lambda\langle Bu, v\rangle = \langle (A + \lambda B)u, v\rangle$$

using properties of the inner product. Therefore, $A^* + \overline{\lambda} B^* = (A + \lambda B)^*$.
(iii) Here again, we have

$$\langle u, B^*(A^*v)\rangle = \langle Bu, A^*v\rangle = \langle (AB)u, v\rangle$$

for all $u, v \in \mathcal{H}$, implying $(AB)^* = B^*A^*$.
(iv) We already know $\|A^*\| \leq \|A\|$. On the other hand, the same inequality with A^* instead of A provides

$$\|(A^*)^*\| \leq \|A^*\|$$

so by (i) we get in fact $\|A\| \leq \|A^*\|$. Henceforth, $\|A^*\| = \|A\|$.

For the last claim, let $u \in \mathcal{H}$ with $\|u\| = 1$. The definition of the operator norm provides

$$\|A^*Au\| \leq \|A^*\|\|Au\| \leq \|A^*\|\|A\|\|u\| = \|A\|^2$$

using $\|A^*\| = \|A\|$ in the last step. On the other hand, an application of Cauchy-Schwarz inequality shows that

$$\|Au\|^2 = \langle Au, Au\rangle = \langle u, A^*Au\rangle \leq \|A^*Au\| \leq \|A^*A\|$$

providing the other bound $\|A\|^2 \leq \|A^*A\|$. We deduce that $\|A^*A\| = \|A\|^2$. Applying this equality to A^* rather than A and using that $\|A^*\| = \|A\|$, one obtains also $\|AA^*\| = \|A\|^2$ as wished. □

A unital C^**−algebra* is a unital algebra[(2)] $\mathcal{A}$ over a field K equipped with a norm $\|\cdot\|_{\mathcal{A}}$ and an anti-linear map $\cdot^*\colon \mathcal{A} \longrightarrow \mathcal{A}$, $a \longmapsto a^*$, which satisfy the following properties:

(i) $(1_{\mathcal{A}})^* = 1_{\mathcal{A}}$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$.

(ii) (Involutivity) $(a^*)^* = a$ for all $a \in \mathcal{A}$.

(iii) (Submultiplicativity) $\|ab\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}}\|b\|_{\mathcal{A}}$ for all $a, b \in \mathcal{A}$.

(iv) (C^*−identity) $\|aa^*\|_{\mathcal{A}} = \|a^*a\|_{\mathcal{A}} = \|a\|^2_{\mathcal{A}}$ for all $a \in \mathcal{A}$.

(v) (Completeness) $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is complete.

Note that if $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a C^*−algebra, we necessarily have $\|a^*\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}$ for all $a \in \mathcal{A}$. Indeed, this equality clearly holds if $a = 0$, and if $a \neq 0$ we have

$$\|a\|^2_{\mathcal{A}} = \|aa^*\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}}\|a^*\|_{\mathcal{A}}$$

and dividing through by $\|a\|_{\mathcal{A}} \neq 0$, one gets $\|a\|_{\mathcal{A}} \leq \|a^*\|_{\mathcal{A}}$. Applying this inequality with a^* rather than a provides $\|a^*\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}}$, whence $\|a\|_{\mathcal{A}} = \|a^*\|_{\mathcal{A}}$ for all $a \in \mathcal{A}$. In particular, the involution map is continuous on $\mathcal{A}$, as if $\varepsilon > 0$ is fixed, choose $\delta := \varepsilon > 0$, and note that if $a, b \in \mathcal{A}$ are so that $\|a - b\|_{\mathcal{A}} < \delta$, then

$$\|a^* - b^*\|_{\mathcal{A}} = \|(a - b)^*\|_{\mathcal{A}} = \|a - b\|_{\mathcal{A}} < \delta = \varepsilon.$$

Proposition 1.5, and the previous results, exactly say that $\mathcal{B}(\mathcal{H})$ is a C^*−algebra, for the involution sending any $A \in \mathcal{B}(\mathcal{H})$ to its adjoint $A^* \in \mathcal{B}(\mathcal{H})$.

In the sequel, we will *not* go through the theory of C^*−algebras, and we formulate all our definitions and results in $\mathcal{B}(\mathcal{H})$. However almost all the theory we develop can be, with slight modifications and small additional precautions, formulated in the more general setting of an arbitrary C^*−algebra, in particular the concepts of spectrum, spectral radius, and the functional calculus for self-adjoint elements. For more background and details on this general setup, see *e.g.* [3, section 5.4.1], [6, chapter VIII].

Let us then return to $\mathcal{B}(\mathcal{H})$. Because of the relation characterizing the adjoint of an operator, we have the following equalities.

(2) If K is a field, an *algebra* over K is a K−vector space $\mathcal{A}$ with a bilinear map $\times\colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, so that $(\lambda a) \times (\mu b) = (\lambda\mu)(a \times b)$ for all $a, b \in \mathcal{A}$, $\lambda, \mu \in K$. Furthermore, if there exists an element $1_{\mathcal{A}} \in \mathcal{A}$ so that $1_{\mathcal{A}} \times a = a \times 1_{\mathcal{A}} = a$ for all $a \in \mathcal{A}$, $\mathcal{A}$ is called *unital*.

Lemma 1.6. Let $A \in \mathcal{B}(\mathcal{H})$. Then $\mathrm{Ker}(A^*) = \mathrm{Im}(A)^\perp$ and $\mathrm{Ker}(A) = \mathrm{Im}(A^*)^\perp$. In particular, if A is invertible, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

Proof. First, suppose that $v \in \mathrm{Ker}(A^*)$, and fix $u \in \mathcal{H}$. Then $\langle Au, v\rangle = \langle u, A^*v\rangle = \langle u, 0\rangle = 0$ so $v \in \mathrm{Im}(A)^\perp$. Conversely, if $v \in \mathrm{Im}(A)^\perp$, then we compute that

$$\|A^*v\|^2 = \langle A^*v, A^*v\rangle = \langle AA^*v, v\rangle = 0$$

which implies $A^*v = 0$, so $v \in \mathrm{Ker}(A^*)$. Hence $\mathrm{Ker}(A^*) = \mathrm{Im}(A)^\perp$. Applying this to A^* rather than A and using $(A^*)^* = A$, we deduce the second equality.

For the last claim, assume that A is invertible. We establish that $(A^{-1})^*A^* = A^*(A^{-1})^* = \mathrm{Id}_{\mathcal{H}}$, which shows at the same time that A^* is invertible and that its inverse is $(A^{-1})^*$. To start, note that

$$\langle (A^{-1})^*A^*u, v\rangle = \langle A^*u, A^{-1}v\rangle = \langle u, AA^{-1}v\rangle = \langle u, v\rangle$$

for any $u, v \in \mathcal{H}$. Likewise, $\langle A^*(A^{-1})^*u, v\rangle = \langle (A^{-1})^*u, Av\rangle = \langle u, A^{-1}Av\rangle = \langle u, v\rangle$ for any $u, v \in \mathcal{H}$. Hence, one gets

$$\langle ((A^{-1})^*A^* - \mathrm{Id}_{\mathcal{H}})u, v\rangle = \langle (A^*(A^{-1})^* - \mathrm{Id}_{\mathcal{H}})u, v\rangle = 0$$

for all $u, v \in \mathcal{H}$, and Lemma 1.3 implies that $(A^{-1})^*A^* = A^*(A^{-1})^* = \mathrm{Id}_{\mathcal{H}}$. We conclude that A^* is invertible, with inverse $(A^*)^{-1} = (A^{-1})^*$. □

1.2 Several classes of operators

In this part we introduce different types of operators, and study the properties they carry.

Definition 1.7. We say that an operator $A \in \mathcal{B}(\mathcal{H})$ is

(i) normal if $A^*A = AA^*$.

(ii) unitary if $A^*A = AA^* = \mathrm{Id}_{\mathcal{H}}$.

(iii) self-adjoint (or hermitian) if $A^* = A$.

(iv) positive if $\langle Au, u\rangle \geq 0$ for any $u \in \mathcal{H}$.

(v) isometric if $\langle Au, Av\rangle = \langle u, v\rangle$ for any $u, v \in \mathcal{H}$.

Additionally, we call an operator $A \in \mathcal{B}(\mathcal{H})$ *negative* if $-A$ is positive.

In the sequel, we write $\mathcal{S}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ for the set of self-adjoint operators on $\mathcal{H}$. It is closed in $\mathcal{B}(\mathcal{H})$ for the norm-topology. Indeed if $(A_n)_{n\in\mathbb{N}}$ is a sequence of self-adjoint operators converging to $A \in \mathcal{B}(\mathcal{H})$ in norm, then

$$\begin{aligned}\|A - A^*\| &= \|A - A_n + A_n - A^*\| \\ &\leq \|A - A_n\| + \|A_n^* - A^*\| \\ &= 2\|A - A_n\|\end{aligned}$$

for all $n \in \mathbb{N}$, using the triangle inequality, the fact that $A_n^* = A_n$ for all $n \in \mathbb{N}$ and Proposition 1.5 for the last equality. Since $\|A - A_n\| \longrightarrow 0$ as $n \to \infty$, we see that $\|A - A^*\| = 0$, so $A = A^*$.

Likewise, $\mathcal{U}(\mathcal{H})$ stands for the set of unitary operators. It is a group with respect to composition.

Finally, $\mathcal{B}(\mathcal{H})^+$ will denote the set of positive operators on $\mathcal{H}$, while $\mathcal{P}(\mathcal{H})$[3] will denote the set of positive invertible operators on $\mathcal{H}$.

Let us provide simple examples to illustrate Definition 1.7.

Example 1.8. (i) A bounded linear operator on a finite-dimensional Hilbert space is a matrix A of size $\dim_{\mathbb{C}}(\mathcal{H}) \times \dim_{\mathbb{C}}(\mathcal{H})$ with complex coefficients a_{ij}, $1 \leq i, j \leq \dim_{\mathbb{C}}(\mathcal{H})$. It is straightforward to check that A is self-adjoint if and only if $a_{ij} = \overline{a_{ji}}$ for all $1 \leq i \leq j \leq \dim_{\mathbb{C}}(\mathcal{H})$.

(ii) Let A be the operator on $L^2([0,1])$ defined by $(Au)(t) = tu(t)$, $t \in [0,1]$. First of all, we have

$$\|Au\|_2^2 = \int_0^1 t^2 |u(t)|^2 \, \mathrm{d}t \leq \|u\|_2^2$$

for all $u \in L^2([0,1])$, whence $\|A\| \leq 1$ and A is bounded. Additionally, we have

$$\langle Au, u \rangle = \int_0^1 tu(t)\overline{u(t)} \, \mathrm{d}t = \int_0^1 t|u(t)|^2 \, \mathrm{d}t \geq 0$$

for any $u \in L^2([0,1])$, so A is positive, in particular self-adjoint and normal. However, it is not an isometry since for $u(t) = v(t) = 1$, $t \in [0,1]$, we have $\langle Au, Av \rangle = \frac{1}{3}$ while $\langle u, v \rangle = 1$.

(iii) Consider the *left shift* operator on $\ell^2(\mathbb{N})$, defined by $(Au)_n = u_{n+1}$, $n \in \mathbb{N}$, for all $u = (u_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$. We see that

$$\|Au\|_2^2 = \|u\|_2^2 - |u_0|^2 \leq \|u\|_2^2$$

for all $u \in \ell^2(\mathbb{N})$, so A is bounded and $\|A\| \leq 1$. Moreover the inequality $\|Au\|_2^2 \leq \|u\|_2^2$ is an equality if we choose any $u \in \ell^2(\mathbb{N})$ with $u_0 = 0$, whence in fact $\|A\| = 1$. It is not an isometry since for $u = (1, 1, 0, 0, \dots)$ we have $\|u\|^2 = 2$ while $\langle Au, Au \rangle = \|Au\|^2 = 1$. The computation

$$\langle Au, v \rangle = \sum_{n\in\mathbb{N}} (Au)_n \overline{v_n}$$

[3]In particular, this has not to be confused with $\mathcal{P}_{\mathrm{s}}(\mathcal{H})$, the set of all subsets of $\mathcal{H}$.

$$= \sum_{n \in \mathbb{N}} u_{n+1} \overline{v_n}$$
$$= \sum_{n \geq 1} u_n \overline{v_{n-1}}$$

valid for all $u, v \in \ell^2(\mathbb{N})$, shows that the adjoint A^* of A is the *right shift* on $\ell^2(\mathbb{N})$, defined by $(A^*u)_0 = 0$ and $(A^*u)_n = u_{n-1}$, $n \geq 1$. In particular A is not self-adjoint.

(iv) If $A \in \mathcal{B}(\mathcal{H})$, the operator A^*A is bounded and positive. Indeed, for $u \in \mathcal{H}$, we compute that
$$\langle A^*Au, u\rangle = \langle Au, Au\rangle = \|Au\|^2 \geq 0.$$
The same reasoning shows that AA^* is positive as well. We will see below that in fact all positive operators arise in this form.

(v) If $A \in \mathcal{B}(\mathcal{H})$ is positive and invertible, then A^{-1} is positive as well. Indeed, fix $u \in \mathcal{H}$, and write it as $u = Av$ for some $v \in \mathcal{H}$. Then
$$\langle A^{-1}u, u\rangle = \langle v, Av\rangle = \langle Av, v\rangle$$
and this last inner product is positive since A is positive.

(vi) If $A, B \in \mathcal{B}(\mathcal{H})$ are both positive, and if $AB = BA$, then AB is positive. We refer to [12, theorem 4.6.9], [15, lemma 6.3.4] for a proof of this fact.

(vii) If $A, B \in \mathcal{B}(\mathcal{H})$ are self-adjoint and $AB = BA$, then AB is self-adjoint, as
$$(AB)^* = B^*A^* = BA = AB.$$
In particular, if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, so are all its powers A^n, $n \in \mathbb{N}$.

Let us now reformulate each of these definitions with the norm.

Proposition 1.9. Let $A \in \mathcal{B}(\mathcal{H})$. The following properties hold.

(i) A is normal if and only if $\|Au\| = \|A^*u\|$ for any $u \in \mathcal{H}$.

(ii) A is unitary if and only if $\|Au\| = \|A^*u\| = \|u\|$ for any $u \in \mathcal{H}$.

(iii) A is self-adjoint if and only if $\langle Au, u\rangle \in \mathbb{R}$ for any $u \in \mathcal{H}$.

(iv) A is an isometry if and only if $\|Au\| = \|u\|$ for any $u \in \mathcal{H}$, and if and only if $A^*A = \mathrm{Id}_{\mathcal{H}}$.

Proof. (i) To start, note that
$$\|Au\|^2 - \|A^*u\|^2 = \langle A^*Au, u\rangle - \langle AA^*u, u\rangle = \langle (A^*A - AA^*)u, u\rangle \tag{1}$$
holds for any $u \in \mathcal{H}$. If A is normal, $A^*A - AA^* = 0$ so the right hand side of (1) vanishes, and we get indeed $\|Au\| = \|A^*u\|$ for any $u \in \mathcal{H}$. Conversely, if this equality holds, we deduce
$$\langle (A^*A - AA^*)u, u\rangle = 0$$

for any $u \in \mathcal{H}$. Lemma 1.3 then implies that $A^*A - AA^* = 0$, so A is normal.

(iii) If A is self-adjoint, we compute that

$$\overline{\langle Au, u\rangle} = \langle u, Au\rangle = \langle u, A^*u\rangle = \langle Au, u\rangle$$

so $\langle Au, u\rangle \in \mathbb{R}$ for any $u \in \mathcal{H}$. Conversely, if this condition holds, then

$$\langle A^*u, u\rangle = \langle u, Au\rangle = \overline{\langle Au, u\rangle} = \langle Au, u\rangle$$

so $\langle (A^* - A)u, u\rangle = 0$ for any $u \in \mathcal{H}$, and as above Lemma 1.3 gives the conclusion.

(iv) First of all, suppose that A preserves the inner product. It implies that $\|Au\|^2 = \langle Au, Au\rangle = \langle u, u\rangle = \|u\|^2$ for any $u \in \mathcal{H}$, so A preserves the norm as well. On the other hand, if A preserves the norm and $u, v \in \mathcal{H}$, one has

$$\begin{aligned}\langle Au, Av\rangle &= \frac{1}{4}(\|A(u+v)\|^2 - \|A(u-v)\|^2 + i\|A(u+iv)\|^2 - i\|A(u-iv)\|^2) \\ &= \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2) \\ &= \langle u, v\rangle\end{aligned}$$

for any $u, v \in \mathcal{H}$. The first (and third) equality relies on the so-called *polarization identity*[(4)], valid in any pre-Hilbert space. This shows that A is an isometry, and concludes the proof of the first equivalence. To prove the second, as for (i) we first note that

$$\|Au\|^2 - \|u\|^2 = \langle Au, Au\rangle - \langle u, u\rangle = \langle (A^*A - \mathrm{Id}_{\mathcal{H}})u, u\rangle \tag{2}$$

for any $u \in \mathcal{H}$. If $A^*A = \mathrm{Id}_{\mathcal{H}}$ the right hand side of (2) equals 0, so $\|Au\| = \|u\|$ for any $u \in \mathcal{H}$. For the other direction, if A preserves the norm, we get

$$\langle (A^*A - \mathrm{Id}_{\mathcal{H}})u, u\rangle = 0$$

for any $u \in \mathcal{H}$, and Lemma 1.3 provides $A^*A = \mathrm{Id}_{\mathcal{H}}$.

(ii) This point is a combination of (i) and (iv). □

Note that in Definition 1.7, an isometric operator on $\mathcal{H}$ is not necessarily an isometry of the metric space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$, as defined in Appendix A, because we do not require the operator to be surjective. In fact it is *not* true that a bounded operator preserving the inner product is surjective: consider for instance the right shift on $\ell^2(\mathbb{N})$. It preserves the inner product, but any sequence whose first coordinate is not 0 does not lie in its image.

However, and we will use this in what follows, an invertible isometry is in fact unitary, *i.e.* if $A \in \mathcal{B}(\mathcal{H})$ is invertible and satisfies $A^*A = \mathrm{Id}_{\mathcal{H}}$, then $AA^* = \mathrm{Id}_{\mathcal{H}}$. Indeed, the equality $A^*A = \mathrm{Id}_{\mathcal{H}}$ implies

$$A^*(AA^*)A = (A^*A)(A^*A) = \mathrm{Id}_{\mathcal{H}} = A^*A$$

[(4)] This identity exactly states that $4\langle u, v\rangle = \|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2$, for any $u, v \in \mathcal{H}$. The best way of proving it is to directly expand the right-hand side.

and multiplying this equality from the left with $(A^*)^{-1}$ (which exists because A is invertible, see Lemma 1.6) and from the right with A^{-1} provides $AA^* = \mathrm{Id}_{\mathcal{H}}$, and thus A is unitary. The same reasoning shows that an invertible operator A with $AA^* = \mathrm{Id}_{\mathcal{H}}$ satisfies also $A^*A = \mathrm{Id}_{\mathcal{H}}$.

For normal operators, Proposition 1.2 takes a simpler form.

Corollary 1.10. Let $A \in \mathcal{B}(\mathcal{H})$ be normal. It holds that

(i) $\mathrm{Ker}(A) = \mathrm{Ker}(A^*)$.

(ii) $\mathrm{Im}(A)$ is dense in $\mathcal{H}$ if and only if A is injective.

(iii) A is invertible if and only if there exists $C > 0$ so that $\|Au\| \geq C\|u\|$ for any $u \in \mathcal{H}$.

Proof. (i) Since A is normal, $\|Au\| = \|A^*u\|$ for any $u \in \mathcal{H}$ by the previous proposition, whence $u \in \mathrm{Ker}(A)$ if and only if $u \in \mathrm{Ker}(A^*)$.

(ii) By the orthogonal decomposition theorem ([3, corollary 1.4.6], [13, theorem A.2]), the fact that $(V^\perp)^\perp = \overline{V}$[5] for any subspace $V \subset \mathcal{H}$ and Lemma 1.6, we have

$$\mathcal{H} = \mathrm{Ker}(A^*) \oplus (\mathrm{Ker}(A^*))^\perp = \mathrm{Ker}(A^*) \oplus (\mathrm{Im}(A)^\perp)^\perp = \mathrm{Ker}(A^*) \oplus \overline{\mathrm{Im}(A)}.$$

Point (i) of the present corollary now implies $\mathcal{H} = \mathrm{Ker}(A) \oplus \overline{\mathrm{Im}(A)}$, whence $\mathrm{Im}(A)$ is dense in $\mathcal{H}$ if and only if A is injective.

(iii) If A is invertible, it suffices to consider $C := \frac{1}{\|A^{-1}\|} > 0$ and the inequality holds, as seen in the proof of Proposition 1.2. Conversely, suppose there is $C > 0$ so that

$$\|Au\| \geq C\|u\|$$

for any $u \in \mathcal{H}$. Again by Proposition 1.2 it is enough to show that the image of A is dense in $\mathcal{H}$. By (ii) above, it is the same as proving that A is injective, which is a consequence of the assumption (as seen in the proof of Proposition 1.2). Thus A is invertible and we are done. □

1.3 Resolvent set, spectrum and spectral radius

The concept of spectrum for an operator generalizes that of eigenvalues for a finite dimensional matrix, and is crucial for the study of the different classes introduced above.

[5] To prove this, first note that $(V^\perp)^\perp$ is a closed subset containing V, so it contains also $\overline{V}$. Conversely, fix $v \in (V^\perp)^\perp$ and write it as $v = v_1 + v_2$, $v_1 \in \overline{V}$, $v_2 \in \overline{V}^\perp$ (by [13, theorem A.2]). As $V \subset \overline{V}$, it follows that $\overline{V}^\perp \subset V^\perp$, so $\langle v, v_2 \rangle = 0$. Using linearity of the inner product and $\langle v_1, v_2 \rangle = 0$, this reads as $\langle v_2, v_2 \rangle = 0$. Thus $v_2 = 0$, and then $v = v_1 \in \overline{V}$, proving the inclusion $(V^\perp)^\perp \subset \overline{V}$.

Definition 1.11. Let $A \in \mathcal{B}(\mathcal{H})$.
The resolvent set of A, denoted $\rho(A)$, is defined as

$$\rho(A) := \{\lambda \in \mathbb{C} : A - \lambda \mathrm{Id}_{\mathcal{H}} \text{ is invertible}\}.$$

The spectrum of A is $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

Before looking at examples, we establish basic properties of the spectrum of an operator.

Proposition 1.12. Let $A \in \mathcal{B}(\mathcal{H})$.
If $|\lambda| > \|A\|$, then $\lambda \in \rho(A)$. In particular, $\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$.

Proof. We start by proving the following claim: if $S \in \mathcal{B}(\mathcal{H})$ has $\|S\| < 1$, then $\mathrm{Id}_{\mathcal{H}} - S$ is invertible.

Indeed, suppose $\|S\| < 1$, and let $S_n := \sum_{k=0}^{n} S^k$. Then for $n, m \in \mathbb{N}$, $n \geq m$, one has

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} S^k \right\| \leq \sum_{k=m+1}^{n} \|S\|^k$$

using the triangle inequality and the submultiplicativity of the norm. The right-hand side is the rest of a convergent series, since $\|S\| < 1$. We thus see that $\|S_n - S_m\| \longrightarrow 0$ as $n, m \to \infty$, *i.e.* $(S_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{B}(\mathcal{H})$. The latter being complete, $(S_n)_{n\in\mathbb{N}}$ converges in $\mathcal{B}(\mathcal{H})$, and we call T its limit. We compute then that

$$(\mathrm{Id}_{\mathcal{H}} - S)T = \lim_{n\to\infty} \left(\sum_{k=0}^{n} S^k - \sum_{k=0}^{n} S^{k+1} \right) = \lim_{n\to\infty} (\mathrm{Id}_{\mathcal{H}} - S^{n+1}) = \mathrm{Id}_{\mathcal{H}}$$

and similarly $T(\mathrm{Id}_{\mathcal{H}} - S) = \mathrm{Id}_{\mathcal{H}}$. Hence $\mathrm{Id}_{\mathcal{H}} - S$ is invertible and $(\mathrm{Id}_{\mathcal{H}} - S)^{-1} = T$.

This claim implies directly the proposition, because if $|\lambda| > \|A\|$, then $\|\frac{A}{\lambda}\| < 1$, so $\mathrm{Id}_{\mathcal{H}} - \frac{A}{\lambda}$ is invertible, and thus so is

$$-\lambda\left(\mathrm{Id}_{\mathcal{H}} - \frac{A}{\lambda}\right) = A - \lambda \mathrm{Id}_{\mathcal{H}}.$$

This implies that $\lambda \in \rho(A)$, and that $\sigma(A)$ is contained in the closed disk of radius $\|A\|$ centered at the origin, finishing the proof. □

This result of boundedness for the spectrum motivates the next definition.

Definition 1.13. Let $A \in \mathcal{B}(\mathcal{H})$.
Its spectral radius, denoted $r(A)$, is defined as

$$r(A) := \sup_{\lambda \in \sigma(A)} |\lambda|.$$

In particular from Proposition 1.12 we directly get

$$r(A) \leq \|A\|$$

for any $A \in \mathcal{B}(\mathcal{H})$, *i.e.* the spectrum of $A \in \mathcal{B}(\mathcal{H})$ is a bounded subset of the complex plane.

Note also that, from Lemma 1.6, $A - \lambda \mathrm{Id}_{\mathcal{H}}$ is invertible if and only if $(A - \lambda \mathrm{Id}_{\mathcal{H}})^* = A^* - \overline{\lambda} \mathrm{Id}_{\mathcal{H}}$ is invertible. This equivalence provides

$$\sigma(A^*) = \{\overline{\lambda} : \lambda \in \sigma(A)\}$$

and in particular $r(A) = r(A^*)$.

The next result, coupled with the boundedness of the spectrum, ensures that the latter is always a compact subset of $\mathbb{C}$.

Proposition 1.14. Let $A \in \mathcal{B}(\mathcal{H})$. Then $\sigma(A) \subset \mathbb{C}$ is closed.

Proof. Let $\lambda \in \rho(A)$. For $\mu \in \mathbb{C}$ so that $|\mu - \lambda| < \frac{1}{\|(A - \lambda \mathrm{Id}_{\mathcal{H}})^{-1}\|}$, the operator $(\mu - \lambda)(A - \lambda \mathrm{Id}_{\mathcal{H}})^{-1} - \mathrm{Id}_{\mathcal{H}}$ is invertible by the claim in the proof of Proposition 1.12, and since

$$A - \mu \mathrm{Id}_{\mathcal{H}} = -(A - \lambda \mathrm{Id}_{\mathcal{H}})((\mu - \lambda)(A - \lambda \mathrm{Id}_{\mathcal{H}})^{-1} - \mathrm{Id}_{\mathcal{H}})$$

we deduce that $A - \mu \mathrm{Id}_{\mathcal{H}}$ is invertible. Thus $\rho(A)$ contains the open ball centered at λ of radius $\frac{1}{\|(A - \lambda \mathrm{Id}_{\mathcal{H}})^{-1}\|}$, and as this holds for any $\lambda \in \rho(A)$, it is an open set. Its complement $\sigma(A)$ is therefore closed in $\mathbb{C}$. □

Let us provide a simple example of computation of spectrum.

Example 1.15. Consider the left shift on $\mathcal{H} = \ell^2(\mathbb{N})$, defined as

$$\begin{aligned} A \colon \ell^2(\mathbb{N}) &\longrightarrow \ell^2(\mathbb{N}) \\ u = (u_n)_{n \in \mathbb{N}} &\longmapsto (u_{n+1})_{n \in \mathbb{N}}. \end{aligned}$$

We have seen above (Example 1.8) that A is bounded with $\|A\| = 1$. Now fix $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, and consider $u := (1, \lambda, \lambda^2, \lambda^3, \dots) \neq 0$. Then

$$\|u\|_2^2 = \sum_{n \in \mathbb{N}} |u_n|^2 = \sum_{n \in \mathbb{N}} |\lambda|^{2n} < \infty$$

as $|\lambda| < 1$, so $u \in \ell^2(\mathbb{N})$, and moreover

$$Au = (\lambda, \lambda^2, \lambda^3, \dots) = \lambda u.$$

Thus any complex number in the open unit disk is an eigenvalue of A, and therefore is in the spectrum of A:

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma(A).$$

As $\sigma(A)$ is closed in $\mathbb{C}$, it must also contain the closure of the open unit disk, which is the closed unit disk. Hence $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subset \sigma(A)$. On the other hand, the reverse inclusion holds by Proposition 1.12, as $\|A\| = 1$. Hence we deduce

$$\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

Here is how the spectrum behaves with respect to translation.

Lemma 1.16. Let $A \in \mathcal{B}(\mathcal{H})$, and $\gamma \in \mathbb{C}$. Then one has

$$\sigma(A - \gamma \mathrm{Id}_{\mathcal{H}}) = \sigma(A) - \gamma$$

where $\sigma(A) - \gamma := \{\lambda - \gamma : \lambda \in \sigma(A)\}$.

Proof. Straightforwardly we have the equivalences

$$\begin{aligned} t \in \sigma(A - \gamma \mathrm{Id}_{\mathcal{H}}) &\iff (A - \gamma \mathrm{Id}_{\mathcal{H}}) - t\mathrm{Id}_{\mathcal{H}} \text{ is not invertible} \\ &\iff A - (\gamma + t)\mathrm{Id}_{\mathcal{H}} \text{ is not invertible} \\ &\iff \gamma + t \in \sigma(A) \\ &\iff t \in \sigma(A) - \gamma \end{aligned}$$

whence the conclusion. □

In general, two arbitrary operators do not commute. However, if $A, B \in \mathcal{B}(\mathcal{H})$, the products AB and BA share the same non-zero spectral points.

Proposition 1.17. Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$$

and, in particular, $r(AB) = r(BA)$.

Proof. Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$. We must show that $AB - \lambda \mathrm{Id}_{\mathcal{H}}$ is invertible if and only if $BA - \lambda \mathrm{Id}_{\mathcal{H}}$ is invertible. First, we handle the case $\lambda = 1$. Suppose then that $AB - \mathrm{Id}_{\mathcal{H}}$ is invertible, and write C for its inverse. Then

$$C(AB - \mathrm{Id}_{\mathcal{H}}) = \mathrm{Id}_{\mathcal{H}}, \ (AB - \mathrm{Id}_{\mathcal{H}})C = \mathrm{Id}_{\mathcal{H}}$$

and re-arranging gives $CAB = \mathrm{Id}_{\mathcal{H}} + C = ABC$. It follows that

$$\begin{aligned}(BCA - \mathrm{Id}_{\mathcal{H}})(BA - \mathrm{Id}_{\mathcal{H}}) &= BCABA - BCA - BA + \mathrm{Id}_{\mathcal{H}} \\ &= B(\mathrm{Id}_{\mathcal{H}} + C)A - BCA - BA + \mathrm{Id}_{\mathcal{H}} \\ &= BA + BCA - BCA - BA + \mathrm{Id}_{\mathcal{H}} \\ &= \mathrm{Id}_{\mathcal{H}}\end{aligned}$$

and also

$$\begin{aligned}(BA - \mathrm{Id}_{\mathcal{H}})(BCA - \mathrm{Id}_{\mathcal{H}}) &= BABCA - BA - BCA + \mathrm{Id}_{\mathcal{H}} \\ &= B(\mathrm{Id}_{\mathcal{H}} + C)A - BA - BCA + \mathrm{Id}_{\mathcal{H}} \\ &= BA + BCA - BA - BCA + \mathrm{Id}_{\mathcal{H}} \\ &= \mathrm{Id}_{\mathcal{H}}\end{aligned}$$

proving that $BA - \mathrm{Id}_{\mathcal{H}}$ is invertible, of inverse

$$(BA - \mathrm{Id}_{\mathcal{H}})^{-1} = B(AB - \mathrm{Id}_{\mathcal{H}})^{-1}A - \mathrm{Id}_{\mathcal{H}}.$$

Exchanging the role of A and B, we get the reverse implication. We now extend to arbitrary $\lambda \neq 0$, using that any scalar commutes with A, B and the particular case we just proved. Indeed

$$\begin{aligned}AB - \lambda\mathrm{Id}_{\mathcal{H}} \text{ is invertible} &\iff \left(\frac{1}{\lambda}A\right)B - \mathrm{Id}_{\mathcal{H}} \text{ is invertible} \\ &\iff B\left(\frac{1}{\lambda}A\right) - \mathrm{Id}_{\mathcal{H}} \text{ is invertible} \\ &\iff (BA - \lambda\mathrm{Id}_{\mathcal{H}})\frac{1}{\lambda} \text{ is invertible} \\ &\iff BA - \lambda\mathrm{Id}_{\mathcal{H}} \text{ is invertible.}\end{aligned}$$

Thus $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ and it immediately follows that $r(AB) = r(BA)$. □

Now we provide alternative descriptions of points in the resolvent set, when the operator in question is self-adjoint.

Lemma 1.18. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and $\lambda \in \mathbb{C}$. The following are equivalent.

(i) $\lambda \in \rho(A)$.

(ii) There exists $C > 0$ so that $\|(A - \lambda\mathrm{Id}_{\mathcal{H}})u\| \geq C\|u\|$ for any $u \in \mathcal{H}$.

Proof. Suppose A is self-adjoint, and $\lambda \in \mathbb{C}$. Then

$$(A - \lambda\mathrm{Id}_{\mathcal{H}})^*(A - \lambda\mathrm{Id}_{\mathcal{H}}) = (A - \overline{\lambda}\mathrm{Id}_{\mathcal{H}})(A - \lambda\mathrm{Id}_{\mathcal{H}})$$

$$= A^2 - \lambda A\mathrm{Id}_{\mathcal{H}} - \overline{\lambda}\mathrm{Id}_{\mathcal{H}}A - \overline{\lambda}\lambda\mathrm{Id}_{\mathcal{H}}$$
$$= (A - \lambda\mathrm{Id}_{\mathcal{H}})(A - \lambda\mathrm{Id}_{\mathcal{H}})^*$$

so $A - \lambda\mathrm{Id}_{\mathcal{H}}$ is normal. Now Corollary 1.10(iii) and the definition of $\rho(A)$ gives the desired equivalence. □

Equivalently, $\lambda \in \sigma(A)$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ so that $\|u_n\| = 1$ for any $n \in \mathbb{N}$ and $\|(A - \lambda\mathrm{Id}_{\mathcal{H}})u_n\| \longrightarrow 0$ as $n \to \infty$.

This characterization allows us to get informations about the localization of the spectrum of a self-adjoint or positive bounded operator.

Proposition 1.19. Let $A \in \mathcal{B}(\mathcal{H})$. The following hold.

(i) If A is self-adjoint, then $\sigma(A) \subset \mathbb{R}$.

(ii) If A is positive, then $\sigma(A) \subset [0, \infty)$.

Proof. (i) Suppose that A is self-adjoint. Let $\lambda = a + ib$ be a complex number with $b \neq 0$. Then one has

$$\begin{aligned}\|(A - \lambda\mathrm{Id}_{\mathcal{H}})u\|^2 &= \langle (A - \lambda\mathrm{Id}_{\mathcal{H}})u, (A - \lambda\mathrm{Id}_{\mathcal{H}})u\rangle \\ &= \|Au\|^2 - (\lambda + \overline{\lambda})\langle Au, u\rangle + |\lambda|^2\|u\|^2 \\ &= \|Au\|^2 - 2a\langle Au, u\rangle + a^2\|u\|^2 + b^2\|u\|^2 \\ &= \|Au\|^2 - 2\mathrm{Re}\langle Au, au\rangle + \|au\|^2 + b^2\|u\|^2 \\ &= \|Au + au\|^2 + b^2\|u\|^2 \\ &\geq b^2\|u\|^2\end{aligned}$$

for any $u \in \mathcal{H}$. Lemma 1.18 then implies $\lambda \in \rho(A)$. Thus $\mathbb{C} \setminus \mathbb{R} \subset \rho(A) = \mathbb{C} \setminus \sigma(A)$, so $\sigma(A) \subset \mathbb{R}$.

(ii) For the case of positive operators, we proceed in the same way. It just remains to exclude negative numbers from the spectrum. Let then $\lambda = a < 0$. Note that $-2a\langle Au, u\rangle \geq 0$ for any $u \in \mathcal{H}$ by positivity of A. Hence with the above computation, we also have

$$\|(A - \lambda\mathrm{Id}_{\mathcal{H}})u\|^2 = \underbrace{\|Au\|^2 - 2a\langle Au, u\rangle}_{\geq 0} + a^2\|u\|^2 \geq a^2\|u\|^2$$

for all $u \in \mathcal{H}$. Invoking again Lemma 1.18, we see that $\rho(A)$ also contains $(-\infty, 0)$, and therefore $\sigma(A) \subset [0, \infty)$. □

Remark 1.20. From the definition of the spectrum, an operator $A \in \mathcal{B}(\mathcal{H})$ is invertible if and only $0 \notin \sigma(A)$. It follows that, if A is positive and invertible, then $\sigma(A) \subset (0, \infty)$. Additionally, since $\sigma(A)$ is closed, it cannot contain points arbitrary close to 0, so there must exist $\varepsilon > 0$ so that $\sigma(A) \subset (\varepsilon, \infty)$.

We now aim at relating this property to inner products of the form $\langle Au, u\rangle$, $u \in \mathcal{H}$.

Definition 1.21. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint.
The numbers

$$m := \inf_{\|u\|=1} \langle Au, u\rangle, \; M := \sup_{\|u\|=1} \langle Au, u\rangle$$

are called respectively the lower bound and the upper bound of A.

It immediately follows from [6, theorem 2.2.13], [13, theorem 1.12] that

$$\|A\| = \max(|m|, |M|)$$

if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint.

Proposition 1.22. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then $m, M \in \sigma(A)$.

Proof. To start, suppose A is positive. In this case, we have $\|A\| = \sup_{\|u\|=1} \langle Au, u\rangle = M \geq m \geq 0$. By definition of M, there is a sequence $(u_n)_{n\geq 1} \subset \mathcal{H}$ so that

$$\|u_n\| = 1, \; \langle Au_n, u_n\rangle \geq M - \frac{1}{n}.$$

for any $n \geq 1$. It follows that

$$\begin{aligned} \|(A - M\mathrm{Id}_{\mathcal{H}})u_n\|^2 &= \|Au_n\|^2 - 2M\langle Au_n, u_n\rangle + M^2\|u_n\|^2 \\ &\leq 2M^2 - 2M\langle Au_n, u_n\rangle \\ &\longrightarrow 0 \end{aligned}$$

as $n \to \infty$, and the remark right after Lemma 1.18 then yields $M \in \sigma(A)$. The same arguing shows that $m \in \sigma(A)$ if A is negative. The general case is handled as follows. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and $\gamma > \max(|m|, |M|) = \|A\|$. Observe that $A + \gamma\mathrm{Id}_{\mathcal{H}}$ is then a positive operator, as

$$\langle (A + \gamma\mathrm{Id}_{\mathcal{H}})u, u\rangle = \langle Au, u\rangle + \gamma\|u\|^2 \geq -\gamma\|u\|^2 + \gamma\|u\|^2 = 0$$

for any $u \in \mathcal{H}$, using the Cauchy-Schwarz inequality. Additionally, observe that $M + \gamma$ is the upper bound of $A + \gamma\mathrm{Id}_{\mathcal{H}}$, whence $M + \gamma \in \sigma(A + \gamma\mathrm{Id}_{\mathcal{H}})$ by the particular case we proved beforehand. By Lemma 1.16, we deduce $M \in \sigma(A)$. Similarly, we reduce the proof of $m \in \sigma(A)$ to the case of a negative operator by considering $A - \gamma\mathrm{Id}_{\mathcal{H}}$. □

We can then obtain the following corollary.

Corollary 1.23. Let $A \in \mathcal{B}(\mathcal{H})$ be positive. The following are equivalent.

(i) A is invertible.

(ii) There exists $\varepsilon > 0$ so that $\langle Au, u\rangle \geq \varepsilon\|u\|^2$ for any $u \in \mathcal{H}$.

Proof. (ii) $\Longrightarrow$ (i) : By the assumption and the Cauchy-Schwarz inequality, one has

$$\|Au\|\|u\| \geq \langle Au, u\rangle \geq \varepsilon\|u\|^2$$

for any $u \in \mathcal{H} \setminus \{0\}$, and dividing through by $\|u\| \neq 0$ yields $\|Au\| \geq \varepsilon\|u\|$ for all $u \in \mathcal{H} \setminus \{0\}$. As the same inequality holds if $u = 0$, and as A is normal, Corollary 1.10(iii) implies that A is invertible.

(i) $\Longrightarrow$ (ii) : We show rather the contrapositive. Suppose that for any $\varepsilon > 0$, there is $u \in \mathcal{H}$ so that $\langle Au, u\rangle < \varepsilon\|u\|^2$. In particular, for each $n \geq 1$, we find $u_n \in \mathcal{H}$ so that

$$\langle Au_n, u_n\rangle < \frac{1}{n}\|u_n\|^2.$$

This inequality implies that $u_n \neq 0$, and we consider $v_n = \frac{u_n}{\|u_n\|}$, for any $n \geq 1$. We have thus found a sequence $(v_n)_{n\geq 1}$ so that $\|v_n\| = 1$ and $\langle Av_n, v_n\rangle < \frac{1}{n}$ for any $n \geq 1$. This easily implies that the lower bound of A is 0, and applying Proposition 1.22, we get that $0 \in \sigma(A)$, whence A is not invertible. □

Here is an important fact on the spectral radius. It is sometimes referred as the *Gelfand's formula*, or the *spectral radius formula*. The proof relies on complex analysis, which we did not introduce, so we will omit it.

Theorem 1.24. Let $A \in \mathcal{B}(\mathcal{H})$. Then $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$.

Proof. See [3, theorem 5.2.7]. □

We derive from this result the following consequences.

Corollary 1.25. Let $A, B \in \mathcal{B}(\mathcal{H})$.

(i) If $AB = BA$, then $r(AB) \leq r(A)r(B)$.

(ii) If A is normal, then $r(A) = \|A\|$.

Proof. (i) Since A and B commute, $(AB)^n = A^n B^n$ for all $n \in \mathbb{N}$. Using Theorem 1.24 and the submultiplicativity of the norm, it follows that

$$r(AB) = \lim_{n\to\infty} \|(AB)^n\|^{1/n} = \lim_{n\to\infty} \|A^n B^n\|^{1/n} \leq \lim_{n\to\infty} \|A^n\|^{1/n} \lim_{n\to\infty} \|B^n\|^{1/n} = r(A)r(B)$$

as announced.

(ii) Suppose first that A is self-adjoint. Using Proposition 1.5(iv), one has $\|A^2\| = \|A\|^2$. Applying this equality with A^2, which is also self-adjoint, yields $\|A^4\| = \|A\|^4$. Continuing this inductive process we get

$$\|A^{2^n}\| = \|A\|^{2^n}$$

for all $n \in \mathbb{N}$. Then $\|A^{2^n}\|^{1/2^n} = \|A\|$ for all $n \in \mathbb{N}$, and we get

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^{2^n}\|^{1/2^n} = \|A\|$$

establishing (ii) in this particular case.

Now, more generally, suppose that A is normal. Applying what we just showed to the operator A^*A, which is self-adjoint, we have

$$\|A\|^2 = \|A^*A\| = r(A^*A) \leq r(A^*)r(A) = r(A)^2 \leq \|A\|^2.$$

Here the first equality is again Proposition 1.5(iv) and the first upper bound is point (i) of the present corollary. The last equality and the last upper bound are the two remarks following Definition 1.13. Hence we deduce $r(A)^2 = \|A\|^2$, giving $r(A) = \|A\|$ and finishing the proof. □

1.4 Functional calculus for self-adjoint operators

The goal of this part is to give a sense to expressions of the form $f(A)$, where A is a self-adjoint operator on $\mathcal{H}$ and f is a continuous function on $\sigma(A) \subset \mathbb{R}$.

To start, if $A \in \mathcal{B}(\mathcal{H})$ and if $P(X) = a_n X^n + \cdots + a_1 X + a_0 \in \mathbb{C}[X]$ is a polynomial in the formal variable X, we define

$$P(A) := a_n A^n + \cdots + a_1 A + a_0 \mathrm{Id}_{\mathcal{H}} \in \mathcal{B}(\mathcal{H}).$$

The first result we establish is that this construction is compatible with the spectrum of an operator. To show this, we need a lemma about commutativity and invertibility of operators.

Lemma 1.26. Let $A, B \in \mathcal{B}(\mathcal{H})$ be so that $AB = BA$. If AB is invertible, then A and B are invertible. More generally, if $A_1 \cdots A_k$ is invertible and $A_1, \ldots, A_k \in$

$\mathcal{B}(\mathcal{H})$ are pairwise commuting, *i.e.*

$$A_i A_j = A_j A_i$$

for all $i, j = 1, \dots, k$, then A_i is invertible for all $i = 1, \dots, k$.

Proof. By assumption, there is $C \in \mathcal{B}(\mathcal{H})$ so that $C(AB) = (AB)C = \mathrm{Id}_{\mathcal{H}}$. By associativity, A has BC as right inverse, and by commutativity of A and B, CB as left inverse. But then

$$(CB)A = \mathrm{Id}_{\mathcal{H}} \Longrightarrow (CBA)BC = BC \Longrightarrow CB(ABC) = BC \Longrightarrow CB = BC.$$

We conclude that BC is the inverse of A. The same reasoning shows that $AC = CA$ is the inverse of B.

Let us now turn our attention to the second statement, that we prove by induction on $k \geq 1$. If $k = 1$, there is nothing to show, and we have just handled the case $k = 2$. If now the statement holds for some $k \geq 1$, and that $A_1, \dots, A_k, A_{k+1} \in \mathcal{B}(\mathcal{H})$ are pairwise commuting with $A_1 \cdots A_{k+1}$ being invertible, then the operators $A_1 \cdots A_k$ and A_{k+1} commute, and their product is invertible. By the case we just proved, we deduce that $A_1 \cdots A_k$ and A_{k+1} are invertible. Now $A_1, \dots, A_k$ are pairwise commuting and their product is invertible, so the induction hypothesis applies, and $A_1, \dots, A_k$ are invertible. This concludes the inductive step, and our proof. □

Proposition 1.27. Let $A \in \mathcal{B}(\mathcal{H})$, $P(X) \in \mathbb{C}[X]$. Then

$$\sigma(P(A)) = P(\sigma(A)) = \{P(\lambda) : \lambda \in \sigma(A)\}.$$

Proof. If P is the zero polynomial or is constant, the claim is obvious. Suppose then that $\deg(P) \geq 1$. First, suppose that $\lambda \notin P(\sigma(A))$. Since $\mathbb{C}$ is algebraically closed, we can factor

$$P(X) - \lambda = c(X - \mu_1) \dots (X - \mu_n)$$

where $n := \deg(P)$, $\mu_1, \dots, \mu_n \in \mathbb{C}$, and $c \in \mathbb{C} \setminus \{0\}$. We deduce that

$$P(\mu_1) = \cdots = P(\mu_n) = \lambda$$

and by assumption $\lambda \notin P(\sigma(A))$, so it follows that $\mu_1, \dots, \mu_n \notin \sigma(A)$. This implies that $A - \mu_i \mathrm{Id}_{\mathcal{H}}$ is invertible for all $i = 1, \dots, n$, and thus so is the operator

$$c(A - \mu_1 \mathrm{Id}_{\mathcal{H}}) \dots (A - \mu_n \mathrm{Id}_{\mathcal{H}}) = P(A) - \lambda \mathrm{Id}_{\mathcal{H}}.$$

Hence $\lambda \notin \sigma(P(A))$, which shows that $\sigma(P(A)) \subset P(\sigma(A))$.

Conversely, suppose $\lambda \in P(\sigma(A))$, and as above write

$$P(X) - \lambda = c(X - \mu_1) \dots (X - \mu_n).$$

We have $\lambda = P(z)$ for some $z \in \sigma(A)$, so z is a root of $P(X) - \lambda$. Up to re-labeling, say $z = \mu_1$. We deduce that $A - \mu_1 \mathrm{Id}_{\mathcal{H}}$ is not invertible, whence

$$c(A - \mu_1 \mathrm{Id}_{\mathcal{H}}) \dots (A - \mu_n \mathrm{Id}_{\mathcal{H}}) = P(A) - \lambda \mathrm{Id}_{\mathcal{H}}$$

is not invertible either, by the contrapositive of Lemma 1.26. We conclude that λ is in $\sigma(P(A))$, as wanted. This concludes the proof. □

Remark 1.28. Note that this result provides an immediate proof of Lemma 1.16.

If A is a bounded self-adjoint operator on $\mathcal{H}$, its spectrum is a compact subset of the real line, and we can consider $C(\sigma(A))$ the Banach space of continuous functions defined on $\sigma(A) \subset \mathbb{R}$ with values in $\mathbb{C}$, equipped with the supremum norm $\|\cdot\|_\infty$. Moreover, we endow this space with a multiplicative structure by setting

$$(fg)(\lambda) := f(\lambda)g(\lambda)$$

for any $f, g \in C(\sigma(A))$, $\lambda \in \sigma(A)$, and an involutive structure via $\overline{f}(\lambda) := \overline{f(\lambda)}$, $f \in C(\sigma(A))$, $\lambda \in \sigma(A)$. Equipped with these maps, $C(\sigma(A))$ is a C^*−algebra (see *e.g.* [6, example 9.1.4]).

The main idea to define $f(A)$ for any self-adjoint operator A and $f \in C(\sigma(A))$ is to approximate f by polynomials appealing the *Weierstrass approximation theorem* and using that any continuous linear map defined on a dense subspace of a normed space and taking values in a Banach space can be uniquely extended to the whole space.

Proposition 1.29. Let X be a normed space, Y be a Banach space, and $E \subset X$ be a dense subspace of X. Denote $\iota\colon E \hookrightarrow X$ the natural injection.
For any continous linear map $f\colon E \longrightarrow Y$, there exists a unique continuous linear map $\tilde{f}\colon X \longrightarrow Y$ so that $\tilde{f} \circ \iota = f$.

Proof. Let $f\colon E \longrightarrow Y$ be linear and continuous, and let $F(t) = \|f\|t$, $t \geq 0$, be a modulus of continuity[(6)] for f. Let $x \in X$. By density of E, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in E so that $x_n \to x$ as $n \to \infty$. In particular $(x_n)_{n\in\mathbb{N}}$ is Cauchy, and as f is linear continuous, $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in Y. As Y is complete, this sequence converges to some $y \in Y$. Set then $\tilde{f}(x) := y$. A priori, this definition of $\tilde{f}(x)$ may depend on the choice of the sequence we make to approach x. If we pick another sequence $(x'_n)_{n\in\mathbb{N}}$ converging to x, then

$$\lim_{n\to\infty} \|x_n - x'_n\| = \|\lim_{n\to\infty}(x_n - x'_n)\| = 0$$

whence $\lim_{n\to\infty} \|f(x_n) - f(x'_n)\| \leq \lim_{n\to\infty} F(\|x_n - x'_n\|) = 0$, proving that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x'_n).$$

(6) Note that any continuous linear map $f\colon X \longrightarrow Y$ between two normed spaces is in fact uniformly continuous, and that $F(t) = \|f\|t$, $t \geq 0$, is a modulus of continuity for f, as in Proposition A.11.

Hence $\tilde{f}$ is well-defined. By construction it is also linear and satisfies $\tilde{f} \circ \iota = f$. Moreover, if $x, y \in X$, choose two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in E converging to x and y respectively, and observe that

$$\begin{aligned} \|\tilde{f}(x) - \tilde{f}(y)\| &= \lim_{n\to\infty} \|f(x_n) - f(y_n)\| \\ &\leq \lim_{n\to\infty} F(\|x_n - y_n\|) \\ &= \|f\| \lim_{n\to\infty} \|x_n - y_n\| \\ &= F(\|x - y\|) \end{aligned}$$

by continuity of the norm. We deduce that $\tilde{f}$ is uniformly continuous, with the same modulus of continuity as f. Additionally, $\tilde{f}$ is unique because continuous and equals to f on the dense subspace $E \subset X$. This finishes the proof. □

As advertised, here is the second main ingredient we need.

Weierstrass approximation theorem. Let $X \subset \mathbb{R}$ be a compact set. Then the subspace of $C(X)$ consisting of polynomial functions is dense in $C(X)$ for the supremum norm $\|\cdot\|_\infty$.

There is in fact a more general version of this result, called the *Stone-Weierstrass theorem*, giving necessary and sufficient conditions for a subalgebra of $C(X)$ to be dense in $C(X)$. We will not show any of these results, and we refer to [3, theorem 5.4.5], [5, theorem II.1.8] or [6, theorem 8.1] for more background and proofs of these theorems.

Henceforth, we are in a position to show the existence of a functional calculus for self-adjoint operators.

Theorem 1.30. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint.
There exists a unique continuous map

$$\begin{aligned} \tilde{\varphi}_A \colon C(\sigma(A)) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ f &\longmapsto f(A) \end{aligned}$$

so that $f(A)$ has its usual sense if f is a polynomial and so that, for all $f, g \in C(\sigma(A))$, $a, b \in \mathbb{C}$, one has

(i) $\|f(A)\| = \|f\|_\infty$.

(ii) $(af + bg)(A) = af(A) + bg(A)$.

(iii) $(fg)(A) = f(A)g(A)$.

(iv) $\overline{f}(A) = f(A)^*$.

(v) $f(A)$ is normal.

(vi) $Bf(A) = f(A)B$ if $B \in \mathcal{B}(\mathcal{H})$ satisfies $AB = BA$.

Proof. By Proposition 1.19, $\sigma(A)$ is a compact subset of $\mathbb{R}$, and by the Weierstrass approximation theorem, the subset D of $C(\sigma(A))$ consisting of polynomials with complex coefficients is dense in $C(\sigma(A))$, for the topology induced by $\|\cdot\|_\infty$. If $P(X) = a_nX^n + \cdots + a_1X + a_0$ is such a polynomial, $P(A) = a_nA^n + \cdots + a_1A + a_0\mathrm{Id}_{\mathcal{H}}$, and as A is self-adjoint and $(A^k)^* = (A^*)^k$ for all $k \in \mathbb{N}$[7], we get

$$P(A)^* = \overline{a_n}A^n + \cdots + \overline{a_1}A + \overline{a_0}\mathrm{Id}_{\mathcal{H}} = \overline{P}(A).$$

In particular, $P(A)P(A)^* = P(A)^*P(A)$, so $P(A)$ is normal. Next, if $B \in \mathcal{B}(\mathcal{H})$ commutes with A, it also commutes with all powers of A, and hence with $P(A)$. Likewise (ii) and (iii) are easily checked if f, g are polynomials. Additionally, it holds that

$$\|P(A)\| = r(P(A)) = \sup_{\lambda\in\sigma(P(A))} |\lambda| = \sup_{\mu\in\sigma(A)} |P(\mu)| = \|P\|_\infty$$

using Corollary 1.25(ii) for the first equality, the definition of the spectral radius for the second, and Proposition 1.27 for the third one. Hence the map

$$\begin{aligned} \varphi_A\colon D &\longrightarrow \mathcal{B}(\mathcal{H}) \\ P &\longmapsto P(A) \end{aligned}$$

is well-defined, and isometric. In particular, it is continuous. It is also linear. The set D being dense in $C(\sigma(A))$, and $\mathcal{B}(\mathcal{H})$ being a Banach space, Proposition 1.29 shows that φ extends uniquely to a continuous linear map

$$\begin{aligned} \tilde{\varphi}_A\colon C(\sigma(A)) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ f &\longmapsto f(A). \end{aligned}$$

The fact that $\tilde{\varphi}_A$ extends φ_A precisely means that $\tilde{\varphi}_A(f) = f(A)$ has its usual sense if f is a polynomial. This proves the first claim. We checked (i)-(vi) are satisfied if f is a polynomial, and by continuity they must remain true for all $f, g \in C(\sigma(A))$. We show this for (i) and (iv) to illustrate how it works, and the other proofs are similar (see *e.g.* [3, theorem 5.4.7]).

Let $f \in C(\sigma(A))$. There exists a sequence $(P_n)_{n\in\mathbb{N}} \subset D$ that converges to f in norm, *i.e.*

$$\lim_{n\to\infty} \|P_n - f\|_\infty = 0.$$

Since $\tilde{\varphi}$ is continuous, we also have $\lim\limits_{n\to\infty} \|f(A) - P_n(A)\|_{\mathcal{B}(\mathcal{H})} = 0$. This implies

$$\begin{aligned} \left|\|f(A)\|_{\mathcal{B}(\mathcal{H})} - \|f\|_\infty\right| &= \left|\|f(A)\|_{\mathcal{B}(\mathcal{H})} - \|P_n(A)\|_{\mathcal{B}(\mathcal{H})} + \|P_n(A)\|_{\mathcal{B}(\mathcal{H})} - \|f\|_\infty\right| \\ &\le \left|\|f(A)\|_{\mathcal{B}(\mathcal{H})} - \|P_n(A)\|_{\mathcal{B}(\mathcal{H})}\right| + \left|\|P_n\|_\infty - \|f\|_\infty\right| \\ &\le \|f(A) - P_n(A)\|_{\mathcal{B}(\mathcal{H})} + \|P_n - f\|_\infty \end{aligned}$$

[7] This is checked by induction on $k \in \mathbb{N}$. For $k = 0$ it reduces to $\mathrm{Id}_{\mathcal{H}} = \mathrm{Id}_{\mathcal{H}}$, and if it holds for $k \in \mathbb{N}$, then

$$(A^{k+1})^* = (A^kA)^* = A^*(A^k)^* = A^*(A^*)^k = (A^*)^{k+1}$$

as announced.

for any $n \in \mathbb{N}$, using several times the two triangle inequalities, and the fact that (i) holds for polynomials. In this last estimate, both terms tend to 0 as $n \to \infty$, and we conclude that

$$\|f(A)\|_{\mathcal{B}(\mathcal{H})} = \|f\|_\infty$$

which shows (i).

Now we turn to (iv). Again let $f \in C(\sigma(A))$, and pick a sequence $(P_n)_{n\in\mathbb{N}}$ in D converging to f in norm. The complex conjugation is the involution of the C^*−algebra $C(\sigma(A))$ and is therefore continuous[8], whence $(\overline{P_n})_{n\in\mathbb{N}}$ converges to $\overline{f}$. By continuity of $\tilde{\varphi}_A$, $(P_n(A))_{n\in\mathbb{N}}$ converges to $f(A)$ and $(\overline{P_n}(A))_{n\in\mathbb{N}}$ converges to $\overline{f}(A)$. Hence, if $u, v \in \mathcal{H}$, we compute that

$$\begin{aligned} \langle u, \overline{f}(A)v\rangle &= \lim_{n\to\infty} \langle u, \overline{P_n}(A)v\rangle \\ &= \lim_{n\to\infty} \langle u, P_n(A)^* v\rangle \\ &= \lim_{n\to\infty} \langle P_n(A)u, v\rangle \\ &= \langle \lim_{n\to\infty} P_n(A)u, v\rangle \\ &= \langle f(A)u, v\rangle \end{aligned}$$

for any $n \in \mathbb{N}$, using continuity of the inner product in each variable for the first and fourth equality, and the fact that (iv) holds for P_n, $n \in \mathbb{N}$. Hence we conclude that $\overline{f}(A) = f(A)^*$, and (iv) is verified for any $f \in C(\sigma(A))$. □

We will refer to property (iii) above by saying that $\tilde{\varphi}_A$ is *multiplicative*.

More generally, if $\psi\colon \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is a map between two C^*−algebras, ψ is called *multiplicative* if it preserves multiplicative structures:

$$\psi(ab) = \psi(a)\psi(b)$$

for any $a, b \in \mathcal{A}_1$.

We note directly from Theorem 1.30(iv) that $f(A)$ is self-adjoint if and only f is real-valued.

Remark 1.31. If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, the functional calculus for A is compatible with conjugation by a fixed bounded invertible operator, *i.e.* if $B \in \mathrm{Aut}(\mathcal{H})$, and $f \in C(\sigma(A))$, then

$$f(B^{-1}AB) = B^{-1}f(A)B.$$

We first see that the above relation holds if f is a polynomial, because it holds for monomials of the form X^n (by induction for instance) and because of linearity of conjugation. If now $f \in C(\sigma(A))$, choose a sequence $(P_n)_{n\in\mathbb{N}}$ so that $P_n \to f$ as $n \to \infty$. We then write that

$$\|f(B^{-1}AB) - B^{-1}f(A)B\| = \|f(B^{-1}AB) - P_n(B^{-1}AB) + P_n(B^{-1}AB) - B^{-1}f(A)B\|$$

[8] As proved right after Proposition 1.5.

$$\leq \|f(B^{-1}AB) - P_n(B^{-1}AB)\| + \|B^{-1}P_n(A)B - B^{-1}f(A)B\|$$
$$\leq \|f - P_n\|_\infty + \|B^{-1}\|\|f - P_n\|_\infty\|B\|$$

for any $n \in \mathbb{N}$, and $\|f - P_n\|_\infty \longrightarrow 0$ as $n \to \infty$. This implies

$$f(B^{-1}AB) = B^{-1}f(A)B$$

as wanted.

Furthermore, $\tilde{\varphi}_A$ is injective, as if $f, g \in C(\sigma(A))$ satisfy $f(A) = \tilde{\varphi}_A(f) = \tilde{\varphi}_A(g) = g(A)$, points (i) and (ii) of Theorem 1.30 provide

$$\|g - f\|_\infty = \|(g - f)(A)\| = \|g(A) - f(A)\| = 0$$

and thus $g = f$. This observation yields the following corollary, which will be useful when introducing square roots.

Corollary 1.32. Let $A \in \mathcal{B}(\mathcal{H})$. The following equivalences hold.

(i) A is self-adjoint if and only if $\sigma(A) \subset \mathbb{R}$.

(ii) A is unitary if and only if $\sigma(A) \subset \mathbb{S}^1$.

(iii) A is positive if and only if A is self-adjoint and $\sigma(A) \subset [0, \infty)$.

Proof. (i) We have the sequence of equivalences

$$\begin{aligned} A = A^* &\iff \tilde{\varphi}_A(\mathrm{Id}_{\sigma(A)}) = \tilde{\varphi}_A(\mathrm{Id}_{\sigma(A)})^* \\ &\iff \tilde{\varphi}_A(\mathrm{Id}_{\sigma(A)}) = \tilde{\varphi}_A(\overline{\mathrm{Id}_{\sigma(A)}}) \\ &\iff \mathrm{Id}_{\sigma(A)} = \overline{\mathrm{Id}_{\sigma(A)}} \end{aligned}$$

using that $\tilde{\varphi}_A$ is involution-preserving (point (iv) of Theorem 1.30) and injective. Since the last equality means exactly $\sigma(A) \subset \mathbb{R}$, this proves the claim.

(ii) Denoting by $\mathbf{1}_{\sigma(A)}$ the constant function equals to 1 on $\sigma(A)$, we have

$$\begin{aligned} A^*A = \mathrm{Id}_{\mathcal{H}} &\iff \tilde{\varphi}_A(\mathrm{Id}_{\sigma(A)})^*\tilde{\varphi}_A(\mathrm{Id}_{\sigma(A)}) = \tilde{\varphi}_A(\mathbf{1}_{\sigma(A)}) \\ &\iff \tilde{\varphi}_A(\overline{\mathrm{Id}_{\sigma(A)}})\tilde{\varphi}_A(\mathrm{Id}_{\sigma(A)}) = \tilde{\varphi}_A(\mathbf{1}_{\sigma(A)}) \\ &\iff \tilde{\varphi}_A(|\mathrm{Id}_{\sigma(A)}|^2) = \tilde{\varphi}_A(\mathbf{1}_{\sigma(A)}) \\ &\iff |\mathrm{Id}_{\sigma(A)}|^2 = \mathbf{1}_{\sigma(A)} \\ &\iff \forall \lambda \in \sigma(A),\ |\lambda|^2 = 1 \\ &\iff \sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}. \end{aligned}$$

If A is unitary, then $A^*A = \mathrm{Id}_{\mathcal{H}}$, and A has spectrum contained in the unit circle.

Conversely, if A has spectrum contained in the unit circle, $A^*A = \mathrm{Id}_{\mathcal{H}}$ and additionally, A is invertible as $0 \notin \sigma(A)$. These two conditions imply $AA^* = \mathrm{Id}_{\mathcal{H}}$ (as explained right after Proposition 1.9), and A is unitary.

(iii) It follows from Proposition 1.9(iii) that a positive operator A is self-adjoint and from Proposition 1.19 that $\sigma(A) \subset [0, \infty)$.

Conversely, if a self-adjoint operator A has $\sigma(A) \subset [0, \infty)$, the function $f(t) = t^{1/2}$ is well-defined and continuous on $\sigma(A)$. We may then apply Theorem 1.30 to A to get an operator $B = f(A)$ with the property that

$$B^2 = f(A)f(A) = (f^2)(A) = (\mathrm{Id}_{\sigma(A)})(A) = A.$$

Additionally, B is self-adjoint since f takes real values, which allows us to compute

$$\langle Au, u\rangle = \langle B^2u, u\rangle = \langle Bu, Bu\rangle = \|Bu\|^2 \geq 0$$

for any $u \in \mathcal{H}$. This proves that A is positive. □

Our next goal is to boost Proposition 1.27, to extend it to any continuous function on the spectrum of a bounded self-adjoint operator. This requires several steps. The first one is the next lemma.

Lemma 1.33. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and $f \in C(\sigma(A))$. Then

$$\sigma(f(A)) \subset f(\sigma(A)).$$

Proof. Suppose $\lambda \notin f(\sigma(A))$. We show that $\lambda \notin \sigma(f(A))$. By assumption, λ is not in the range of f, so we may consider the function

$$g(z) := \frac{1}{f(z) - \lambda}, \quad z \in \sigma(A)$$

which is also in $C(\sigma(A))$. One has then

$$\begin{aligned}(f(A) - \lambda\mathrm{Id}_{\mathcal{H}})g(A) &= ((f - \lambda)g)(A) = \mathbf{1}_{\sigma(A)}(A) = \mathrm{Id}_{\mathcal{H}}\\ g(A)(f(A) - \lambda\mathrm{Id}_{\mathcal{H}}) &= (g(f - \lambda))(A) = \mathbf{1}_{\sigma(A)}(A) = \mathrm{Id}_{\mathcal{H}}\end{aligned}$$

using point (iii) of Theorem 1.30. This shows that $f(A) - \lambda\mathrm{Id}_{\mathcal{H}}$ is invertible. Thus $\lambda \notin \sigma(f(A))$ and we conclude that $\sigma(f(A)) \subset f(\sigma(A))$. □

To prove the converse, we will appeal the next fact about invertible operators.

Lemma 1.34. $\mathrm{Aut}(\mathcal{H})$ is open in $\mathcal{B}(\mathcal{H})$.

Proof. Fix $A \in \mathrm{Aut}(\mathcal{H})$, and let $B \in \mathcal{B}(\mathcal{H})$ be so that $\|A - B\| < \frac{1}{\|A^{-1}\|}$. Then

$$\|\mathrm{Id}_{\mathcal{H}} - A^{-1}B\| = \|A^{-1}(A - B)\| \leq \|A^{-1}\|\|A - B\| < 1$$

and it follows from the claim proved in Proposition 1.12 that $\mathrm{Id}_{\mathcal{H}} - (\mathrm{Id}_{\mathcal{H}} - A^{-1}B) = A^{-1}B$ is invertible. Thus $B = A(A^{-1}B)$ is invertible as a product of two invertible operators. This shows that $\mathrm{Aut}(\mathcal{H})$ is a neighbourhood of any of its elements, *i.e.* is open in $\mathcal{B}(\mathcal{H})$. □

We are now ready to see that continuous functions preserve spectrums.

Theorem 1.35. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and $f \in C(\sigma(A))$. Then

$$\sigma(f(A)) = f(\sigma(A)).$$

Proof. One inclusion is Lemma 1.33. We prove the reverse inclusion. Let $\lambda \in \sigma(A)$ and set $\mu := f(\lambda)$. We must argue that $\mu \in \sigma(f(A))$. Towards a contradiction, suppose that $\mu \notin \sigma(f(A))$. This implies that $f(A) - \mu\mathrm{Id}_{\mathcal{H}}$ is invertible. Now, choose a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ that converges to f in $C(\sigma(A))$. By continuity of $\tilde{\varphi}_A$, the sequence

$$(P_n(A) - P_n(\lambda)\mathrm{Id}_{\mathcal{H}})_{n\in\mathbb{N}}$$

converges to $f(A) - \mu\mathrm{Id}_{\mathcal{H}}$, and since the latter is invertible, $P_n(A) - P_n(\lambda)\mathrm{Id}_{\mathcal{H}}$ is invertible for n large enough, by Lemma 1.34. Thus $P_n(\lambda) \notin \sigma(P_n(A))$ for n large enough, which is absurd in view of Proposition 1.27. We conclude that $\mu \in \sigma(f(A))$, and the proof is complete. □

This theorem has immediate consequences.

Corollary 1.36. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $f \in C(\sigma(A))$. Then

(i) $f(A)$ is positive if and only if f takes positive real values.

(ii) $f(A)$ is unitary if and only if f takes values in $\mathbb{S}^1$.

Proof. (i) Using Corollary 1.32(iii), $f(A)$ is positive if and only if $f(A)$ is self-adjoint and $\sigma(f(A)) \subset [0,\infty)$, or equivalently f is real-valued and $f(\sigma(A)) \subset [0,\infty)$, by Theorem 1.35. These two conditions are equivalent to the single one $f(\sigma(A)) \subset [0,\infty)$, whence the claim.

(ii) Likewise, by Corollary 1.32(ii), $f(A)$ is unitary if and only if $\sigma(f(A)) \subset \mathbb{S}^1$, *i.e.* if and only if $f(\sigma(A)) \subset \mathbb{S}^1$, by Theorem 1.35. □

As already used several times, if $f \in C(\sigma(A))$ is real-valued, then $f(A)$ is self-adjoint. In that case, it is then possible to apply Theorem 1.30 to $f(A)$ itself. The next proposition describes the functional calculus for $f(A)$ in terms of the one for A.

Proposition 1.37. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and $f \in C(\sigma(A))$ be real-valued. If $g \in C(\sigma(f(A)))$, then $g \circ f \in C(\sigma(A))$ and

$$\tilde{\varphi}_A(g \circ f) = \tilde{\varphi}_{f(A)}(g).$$

Proof. It is a uniqueness arguing. Denote

$$\begin{aligned} \psi : C(\sigma(f(A))) &\longrightarrow C(\sigma(A)) \\ g &\longmapsto g \circ f. \end{aligned}$$

Let $g \in C(\sigma(f(A)))$. Since $\sigma(f(A)) = f(\sigma(A))$ by Theorem 1.35, g is continuous on the image of f, so $g \circ f$ is continuous on $\sigma(A)$ and ψ is well-defined. With this notation, we are left to prove

$$\tilde{\varphi}_A \circ \psi = \tilde{\varphi}_{f(A)}.$$

Note first that ψ is an isometry, as

$$\|\psi(g)\|_\infty = \|g \circ f\|_\infty = \sup_{x \in \sigma(A)} |g \circ f(x)| = \sup_{t \in \sigma(f(A))} |g(t)| = \|g\|_\infty$$

for any $g \in C(\sigma(f(A)))$, where the third equality relies on Theorem 1.35. In particular, ψ is continuous. It is furthermore linear and multiplicative, as

$$\psi(ag_1 + bg_2) = (ag_1 + bg_2) \circ f = a(g_1 \circ f) + b(g_2 \circ f) = a\psi(g_1) + b\psi(g_2)$$

for any $g_1, g_2 \in C(\sigma(f(A)))$, $a, b \in \mathbb{C}$, as well as

$$\psi(g_1 g_2) = (g_1 g_2) \circ f = (g_1 \circ f)(g_2 \circ f) = \psi(g_1)\psi(g_2)$$

for all $g_1, g_2 \in C(\sigma(f(A)))$. This implies that $\tilde{\varphi}_A \circ \psi$ is continuous, linear and multiplicative. Now, we compute that

$$\begin{aligned} \tilde{\varphi}_A \circ \psi(\mathrm{Id}_{\sigma(f(A))}) &= \tilde{\varphi}_A(\psi(\mathrm{Id}_{\sigma(f(A))})) \\ &= \tilde{\varphi}_A(\mathrm{Id}_{\sigma(f(A))} \circ f) \\ &= \tilde{\varphi}_A(f) \\ &= f(A) \\ &= \tilde{\varphi}_{f(A)}(\mathrm{Id}_{\sigma(f(A))}) \end{aligned}$$

meaning that $\tilde{\varphi}_A \circ \psi$ and $\tilde{\varphi}_{f(A)}$ agree on $\mathrm{Id}_{\sigma(f(A))}$. Since both maps are linear and multiplicative, they agree on the dense subset of polynomials in $C(\sigma(f(A)))$. Coupled with continuity of both maps already established, the uniqueness part in Theorem 1.30 forces to have $\tilde{\varphi}_A \circ \psi = \tilde{\varphi}_{f(A)}$, as wanted. We are done. □

Now, if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, Theorem 1.30 provides a new operator $\exp(A) \in \mathcal{B}(\mathcal{H})$, which is self-adjoint since the exponential is real-valued, and in fact positive since the exponential takes positive values (Corollary 1.36(i)). It turns out this process is invertible.

Corollary 1.38. The exponential map $\exp\colon \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{P}(\mathcal{H})$ is a bijection.

Proof. Let $A \in \mathcal{S}(\mathcal{H})$. Then $\exp(A)$ is positive, and also invertible since $\exp(x) > 0$ for any $x \in \sigma(A)$ and $\sigma(\exp(A)) = \exp(\sigma(A))$. The map $\exp\colon \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{P}(\mathcal{H})$ is thus well-defined. Moreover, $\sigma(\exp(A)) \subset (0,\infty)$, so the logarithm is well-defined and continuous on $\sigma(\exp(A))$. By Proposition 1.37 we have

$$\tilde{\varphi}_A(\ln \circ \exp) = \tilde{\varphi}_{\exp(A)}(\ln)$$

i.e. $\ln(\exp(A)) = (\ln \circ \exp)(A)$. The right-hand side reduces to $\mathrm{Id}_{\sigma(A)}(A) = A$, so that

$$\ln(\exp(A)) = A.$$

Conversely, if $A \in \mathcal{P}(\mathcal{H})$ then $\sigma(A) \subset (0,\infty)$, $\ln \in C(\sigma(A))$ and is real-valued, thus $\ln(A)$ is self-adjoint. Applying the exponential yields as above to

$$\exp(\ln(A)) = \mathrm{Id}_{\sigma(A)}(A) = A$$

showing that $\exp\colon \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{P}(\mathcal{H})$ is a bijection, of inverse $\ln\colon \mathcal{P}(\mathcal{H}) \longrightarrow \mathcal{S}(\mathcal{H})$. □

Remark 1.39. (i) Another way of defining the exponential of a self-adjoint operator is the following: for $A \in \mathcal{S}(\mathcal{H})$, consider the sequence of operators

$$S_n := \sum_{k=0}^{n} \frac{A^k}{k!}$$

and observe that, if $n \geq m \geq 0$, then

$$\|S_n - S_m\| = \left\| \sum_{k=0}^{n} \frac{A^k}{k!} - \sum_{k=0}^{m} \frac{A^k}{k!} \right\| = \left\| \sum_{k=m+1}^{n} \frac{A^k}{k!} \right\| \leq \sum_{k=m+1}^{n} \frac{\|A\|^k}{k!}$$

and since the last sum goes to 0 as $n, m \longrightarrow \infty$ (it is the rest of the series that defines the real number $e^{\|A\|}$), the sequence $(S_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{B}(\mathcal{H})$, and therefore converges to a bounded operator, that we define to be $\exp(A)$. In fact, this is *exactly* the way we defined $\exp(A)$ via functional calculus and Theorem 1.30, as its proof and the proof of Proposition 1.29 shows it. This way, we will also work with this explicit construction, invoking the sequence of *partial sums* $(S_n)_{n\in\mathbb{N}}$.

(ii) On the other hand, note that the procedure exposed in (i) works in fact for any bounded operator, not just self-adjoint ones, whereas Theorem 1.30 *is* restricted to self-adjoint operators.

As for real numbers, we will write either $\exp(A)$ or e^A for the exponential of $A \in \mathcal{B}(\mathcal{H})$.

One has the next properties for the exponential of an operator: for $A, B \in \mathcal{B}(\mathcal{H})$, it holds

(i) $e^0 = \mathrm{Id}_{\mathcal{H}}$.

(ii) $(e^A)^* = e^{A^*}$.

(iii) If $AB = BA$, then $e^{A+B} = e^A e^B$.

(iv) $(e^A)^{-1} = e^{-A}$.

(v) $\|e^A\| \leq e^{\|A\|}$.

(i) is by definition, and (ii) follows from the continuity of the involution of $\mathcal{B}(\mathcal{H})$[9]:

$$(e^A)^* = \left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{A^k}{k!} \right)^* = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{A^k}{k!} \right)^* = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(A^*)^k}{k!} = e^{A^*}.$$

(iv) is an immediate consequence of (iii) and (i), with $B = -A$, and (v) follows from the submultiplicativity and the continuity of the norm:

$$\|e^A\| = \left\| \lim_{n\to\infty} \sum_{k=0}^{n} \frac{A^k}{k!} \right\| \leq \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\|A\|^k}{k!} = e^{\|A\|}.$$

To establish (iii), note first of all that as A and B commute, the *binom Newton's formula* holds:

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$

for all $n \in \mathbb{N}$. The proof is readily the same as for real numbers, and can be done by induction on $n \in \mathbb{N}$ for instance. Thus it follows that

$$\begin{aligned} e^A e^B &= \left(\sum_{i=0}^{\infty} \frac{A^i}{i!} \right) \left(\sum_{j=0}^{\infty} \frac{B^j}{j!} \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i+j=k} \frac{A^i}{i!} \frac{B^j}{j!} \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \frac{A^i}{i!} \frac{B^{k-i}}{(k-i)!} \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i} \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (A+B)^k \\ &= e^{A+B} \end{aligned}$$

as wished.

[9] Actually, we also use in the line just below the fact that $(A^k)^* = (A^*)^k$ for any $A \in \mathcal{B}(\mathcal{H})$ and $k \in \mathbb{N}$, as we proved earlier.

Remark 1.40. Using Theorem 1.30 and Proposition 1.37, one can prove similar identities for bounded operators as the ones for real numbers. For instance, if $\alpha, \beta > 0$ and $A \in \mathcal{P}(\mathcal{H})$, then $A^{\alpha+\beta} = A^{\alpha}A^{\beta}$. Likewise, $\ln(A^{\alpha}) = \alpha\ln(A)$, and $A^{\alpha} = \exp(\alpha\ln(A))$.

1.5 Square roots and polar decomposition

The goal here is to introduce square roots of bounded operators. The definition is exactly the same as for positive real numbers.

Definition 1.41. Let $A \in \mathcal{B}(\mathcal{H})$.
A square root of A is a bounded operator B on $\mathcal{H}$ so that $B^2 = A$.

When it exists, the square root of a bounded operator A is denoted $\sqrt{A}$ or $A^{1/2}$.

A sufficient condition to guarantee the existence of a square root for an operator is precisely, as for real numbers, to be positive.

Theorem 1.42. Let $A \in \mathcal{B}(\mathcal{H})$ be positive.
Then A has a unique positive square root B. Moreover, if A is invertible, so is B.

Proof. Suppose that A is positive. Then $\sigma(A) \subset [0,\infty)$, and as in the proof of Corollary 1.32 we consider $f \in C(\sigma(A))$ defined by $f(t) = \sqrt{t}$, $t \in \sigma(A)$. As already seen, $B = f(A)$ is self-adjoint and satisfies $B^2 = A$. Moreover, B is positive by Corollary 1.36(i) and exactly as in Corollary 1.38 we establish that

$$\sqrt{\cdot} : \mathcal{B}(\mathcal{H})^+ \longrightarrow \mathcal{B}(\mathcal{H})^+$$

is a bijection. In particular B is unique. Lastly, assume A is invertible. A commutes with itself, so also with $f(A) = B$ by (vi) of Theorem 1.30. Thus, by the same result, B commutes also with A^{-1} (since $t \longmapsto \frac{1}{t}$ is indeed continuous on $\sigma(A) \subset (0,\infty)$). Hence one has

$$B(A^{-1}B) = A^{-1}B^2 = A^{-1}A = \mathrm{Id}_{\mathcal{H}}, \;\; (A^{-1}B)B = A^{-1}B^2 = A^{-1}A = \mathrm{Id}_{\mathcal{H}}$$

and then B is invertible of inverse $A^{-1}B$. This concludes the proof. □

Here also, the same remark as for the exponential applies: a more hand-by-hand approach for building the square root of a positive operator A, followed for instance in [12, theorem 4.6.14], consists in defining a sequence of polynomials in A and using completeness. This is exactly what we did above, in the proof of Proposition 1.29 and Theorem 1.30.

As promised, we can now complete Example 1.8(iv).

Corollary 1.43. An operator $P \in \mathcal{B}(\mathcal{H})$ is positive if and only there exists $A \in \mathcal{B}(\mathcal{H})$ so that $P = A^*A$.

Proof. One direction is Example 1.8(iv). Conversely, if P is positive, let $A := \sqrt{P}$. Then $A^* = \sqrt{P}$ and $A^*A = \sqrt{P}\sqrt{P} = P$. □

As for the exponential and the logarithm, we outline rules of computations with square roots of bounded operators. They will be particularly useful in Chapter 3.

Corollary 1.44. Let $A, B \in \mathcal{B}(\mathcal{H})^+$. The following hold.

(i) If $AB = BA$, then $\sqrt{AB} = \sqrt{A}\sqrt{B}$.

(ii) If A is invertible, then $\sqrt{A}$ is invertible and $\left(\sqrt{A}\right)^{-1} = \sqrt{A^{-1}}$.

(iii) $\left\|\sqrt{A}\right\| = \sqrt{\|A\|}$.

Proof. (i) If $AB = BA$, applying twice Theorem 1.30(vi) shows that $\sqrt{A}$ commutes with $\sqrt{B}$. Now we observe that

$$\left(\sqrt{A}\sqrt{B}\right)^2 = \sqrt{A}\sqrt{B}\sqrt{A}\sqrt{B} = \sqrt{A}\sqrt{A}\sqrt{B}\sqrt{B} = AB$$

meaning that $\sqrt{A}\sqrt{B}$ is a square root of AB. The uniqueness part of Theorem 1.42 now forces to have $\sqrt{AB} = \sqrt{A}\sqrt{B}$, as announced.

(ii) The invertibility of $\sqrt{A}$ has already been derived in Theorem 1.42, and an explicit formula for its inverse can be found with point (i) of the present corollary (that we may apply since A commutes with its inverse):

$$\sqrt{A^{-1}}\sqrt{A} = \sqrt{A^{-1}A} = \sqrt{\mathrm{Id}_{\mathcal{H}}} = \mathrm{Id}_{\mathcal{H}} = \sqrt{AA^{-1}} = \sqrt{A}\sqrt{A^{-1}}.$$

The uniqueness of the inverse of an operator now ensures $\left(\sqrt{A}\right)^{-1} = \sqrt{A^{-1}}$.

(iii) is a consequence of the C^*–identity in $\mathcal{B}(\mathcal{H})$. Indeed, one has

$$\left\|\sqrt{A}\right\|^2 = \left\|\sqrt{A}\left(\sqrt{A}\right)^*\right\| = \left\|\sqrt{A}\sqrt{A}\right\| = \|A\|$$

whence $\left\|\sqrt{A}\right\| = \sqrt{\|A\|}$. □

Now we can introduce polar decompositions for bounded operators.

Definition 1.45. Let $A \in \mathcal{B}(\mathcal{H})$.

A polar decomposition of A is a factorization as

$$A = PU$$

where U is unitary and P is positive.

It turns out such a decomposition exists for any invertible operator.

Theorem 1.46. Let $A \in \mathrm{Aut}(\mathcal{H})$. Then A has a unique polar decomposition.

Proof. As A is invertible, so are A^* and AA^*. Moreover, AA^* is positive (by Example 1.8(iv)), so it has a square root. Set $P := \sqrt{AA^*}$, which is invertible by Theorem 1.42, and $U := P^{-1}A$. Then P is positive and $A = PU$. Additionally, we have

$$UU^* = P^{-1}AA^*(P^{-1})^* = (AA^*)^{-1/2}AA^*(AA^*)^{-1/2} = \mathrm{Id}_{\mathcal{H}}$$

and likewise

$$U^*U = A^*(P^{-1})^*P^{-1}A = A^*(AA^*)^{-1}A = \mathrm{Id}_{\mathcal{H}}.$$

Thus U is unitary, and this proves the existence of a polar decomposition. For the uniqueness part, suppose that $A = PU = P'U'$ with P, P' positive and $U, U' \in \mathcal{U}(\mathcal{H})$. Then we have

$$AA^* = PUU^*P^* = P^2, \; AA^* = P'U'(U')^*(P')^* = (P')^2$$

so P and P' are both positive square roots of the positive operator AA^*. The uniqueness part of Theorem 1.42 then forces $P' = P$, and thus also

$$U' = (P')^{-1}A = P^{-1}A = U.$$

This concludes the proof. □

1.6 The weak operator topology on $\mathcal{B}(\mathcal{H})$

Let us start the discussion on topologies for $\mathcal{B}(\mathcal{H})$ with the next lemma.

Lemma 1.47. Let $A \in \mathcal{B}(\mathcal{H})$.
The maps $X \longmapsto AX$, $X \longmapsto XA$ are continuous from $(\mathcal{B}(\mathcal{H}), \tau_{\|\cdot\|})$ to $(\mathcal{B}(\mathcal{H}), \tau_{\|\cdot\|})$.

Proof. Fix $A \in \mathcal{B}(\mathcal{H})$. We show the continuity of $X \longmapsto AX$, the other one is very similar. If $A = 0$, the claim is obvious, so we may suppose that $A \neq 0$. Using Theorem A.10, let $\varepsilon > 0$, and set $\delta := \frac{\varepsilon}{\|A\|} > 0$. If $X, Y \in \mathcal{B}(\mathcal{H})$ are so that $\|X - Y\| < \delta$, then

$$\|AX - AY\| \leq \|A\|\|X - Y\| < \|A\|\delta = \varepsilon$$

and the conclusion follows. □

As mentioned earlier, a norm $\|\cdot\|$ on a vector space X provides a metric d_X, and this metric provides thus the structure of a topological space to X. In general, the topology obtained in this way has many open sets, and it is harder to show convergence of sequences, or to get compact sets. However, as seen in Appendix A, compactness for metrisable spaces is equivalent to sequential compactness, and we want to take advantage of this to establish existence of objects with special properties (by extracting a convergent subsequence of a well-chosen sequence generally).

The goal of this subsection is to introduce a new topology on the space $\mathcal{B}(\mathcal{H})$, which is smaller than the norm topology $\tau_{\|\cdot\|}$.

This topology is called the *weak operator* topology, is denoted τ_w, and is the initial topology on $\mathcal{B}(\mathcal{H})$ generated by the family of linear functionals

$$\mathcal{F} := \{\omega_{u,v} \colon \mathcal{B}(\mathcal{H}) \longrightarrow \mathbb{C} \mid u, v \in \mathcal{H}\}$$

where

$$\begin{aligned} \omega_{u,v} \colon \mathcal{B}(\mathcal{H}) &\longrightarrow \mathbb{C} \\ A &\longmapsto \langle Au, v\rangle. \end{aligned}$$

In other words, τ_w is the topology generated by the subbasis

$$\mathcal{B}_{\mathcal{F}} := \{\omega_{u,v}^{-1}(U) : u, v \in \mathcal{H},\ U \subset \mathbb{C} \text{ open}\}$$

and, in particular, sets of the form

$$V(A; u_1, v_1, u_2, v_2, \ldots, u_n, v_n, \varepsilon) = \{B \in \mathcal{B}(\mathcal{H}) : |\langle (A - B)u_i, v_i\rangle| < \varepsilon,\ i = 1, \ldots, n\}$$

are a basis of neighbourhoods for A in τ_w. Furthermore, by Proposition A.39, a sequence $(A_n)_{n\in\mathbb{N}}$ converges to $A \in \mathcal{B}(\mathcal{H})$ in τ_w if and only if $\omega_{u,v}(A_n) \longrightarrow \omega_{u,v}(A)$ as $n \to \infty$, for all $u, v \in \mathcal{H}$, *i.e.* if and only if

$$\langle A_n u, v\rangle \longrightarrow \langle Au, v\rangle$$

as $n \to \infty$, for all $u, v \in \mathcal{H}$. In this case we say that $(A_n)_{n\in\mathbb{N}}$ converges *weakly* to A.

Let us note the following.

Lemma 1.48. The weak operator topology is smaller than the norm topology.

Proof. Let $U \in \tau_w$, and fix $A \in U$. As U is weakly open, there exists $u_1, v_1, \dots, u_n, v_n \in \mathcal{H}$ and $\varepsilon > 0$ so that

$$\bigcap_{i=1}^{n} \{B \in \mathcal{B}(\mathcal{H}) : |\langle (A - B)u_i, v_i \rangle| < \varepsilon\} \subset U.$$

Let then $\delta_i := \frac{\varepsilon}{1+\|v_i\|\|u_i\|} > 0$ for all $i = 1, \dots, n$ and $\delta := \min(\delta_1, \dots, \delta_n) > 0$.

By the Cauchy-Schwarz inequality, if $B \in \mathcal{B}(\mathcal{H})$ is so that $\|A - B\| < \delta$, then

$$\begin{aligned} |\langle (A - B)u_i, v_i \rangle| &\leq \|A - B\|\|u_i\|\|v_i\| \\ &< \delta \|u_i\|\|v_i\| \\ &\leq \delta_i \|u_i\|\|v_i\| \\ &= \frac{\varepsilon \|u_i\|\|v_i\|}{1 + \|u_i\|\|v_i\|} \\ &< \varepsilon \end{aligned}$$

for all $i = 1, \dots, n$, whence the inclusion

$$\{B \in \mathcal{B}(\mathcal{H}) : \|A - B\| < \delta\} \subset \{B \in \mathcal{B}(\mathcal{H}) : |\langle (A - B)u_i, v_i \rangle| < \varepsilon\}$$

for all $i = 1, \dots, n$. Thus it follows that

$$\{B \in \mathcal{B}(\mathcal{H}) : \|A - B\| < \delta\} \subset \bigcap_{i=1}^{n} \{B \in \mathcal{B}(\mathcal{H}) : |\langle (A - B)u_i, v_i \rangle| < \varepsilon\} \subset U$$

and hence U is a neighbourhood of any of its elements in the norm topology. It is therefore in $\tau_{\|\cdot\|}$. This concludes our proof. ◻

For the weak operator topology, left and right multiplication by a bounded operator remain continuous maps.

Lemma 1.49. Let $A \in \mathcal{B}(\mathcal{H})$.
The maps $X \longmapsto AX$, $X \longmapsto XA$ are continuous from $(\mathcal{B}(\mathcal{H}), \tau_w)$ to $(\mathcal{B}(\mathcal{H}), \tau_w)$.

Proof. Let φ be the map sending $X \in \mathcal{B}(\mathcal{H})$ to $AX \in \mathcal{B}(\mathcal{H})$. As τ_w is an initial topology, we can use Proposition A.38 to prove that φ is continuous. It is therefore enough to prove that the composition $\omega_{u,v} \circ \varphi \colon \mathcal{B}(\mathcal{H}) \longrightarrow \mathbb{C}$ is continuous, for any fixed pair of vectors $u, v \in \mathcal{H}$. Let then $u, v \in \mathcal{H}$, and observe that

$$(\omega_{u,v} \circ \varphi)(X) = \omega_{u,v}(AX) = \langle AXu, v \rangle = \langle Xu, A^* v \rangle = \omega_{u,A^* v}(X)$$

for all $X \in \mathcal{B}(\mathcal{H})$. Hence $\omega_{u,v} \circ \varphi = \omega_{u,A^* v}$ which is continuous by definition of τ_w. Thus φ is continuous.

If ψ is the map sending $X \in \mathcal{B}(\mathcal{H})$ to $XA \in \mathcal{B}(\mathcal{H})$, the same reasoning applies, observing this time that

$$(\omega_{u,v} \circ \psi)(X) = \langle XAu, v\rangle = \omega_{Au,v}(X)$$

for all $X \in \mathcal{B}(\mathcal{H})$. Thus $\omega_{u,v} \circ \psi = \omega_{Au,v}$ for any $u, v \in \mathcal{H}$, which is continuous by definition of τ_w. We conclude that ψ is continuous, and the lemma is established. □

As already mentioned, τ_w has less open sets than $\tau_{\|\cdot\|}$. However it still has enough open sets to separate points.

Lemma 1.50. The weak operator topology on $\mathcal{B}(\mathcal{H})$ is Hausdorff.

Proof. We start by proving that the family $\mathcal{F}$ separates points, in the sense that if $A \neq B$ in $\mathcal{B}(\mathcal{H})$, there exists $u, v \in \mathcal{H}$ so that $\omega_{u,v}(A) \neq \omega_{u,v}(B)$. If $A \neq B$, then $A - B \neq 0$, and Lemma 1.3 thus ensures that there is $u, v \in \mathcal{H}$ so that $\langle (A-B)u, v\rangle \neq 0$. This inner product being non-zero exactly means $\omega_{u,v}(A) \neq \omega_{u,v}(B)$, as claimed. Now, if $A \neq B \in \mathcal{B}(\mathcal{H})$, we pick $u, v \in \mathcal{H}$ so that $\omega_{u,v}(A) \neq \omega_{u,v}(B)$, and as $\mathbb{C}$ is Hausdorff, there exists two open sets $U_1, U_2 \subset \mathbb{C}$ with

$$U_1 \cap U_2 = \emptyset,\ \omega_{u,v}(A) \in U_1,\ \omega_{u,v}(B) \in U_2.$$

It follows that $\omega_{u,v}^{-1}(U_1), \omega_{u,v}^{-1}(U_2)$ are disjoint, and weakly open by definition of τ_w. Moreover $A \in \omega_{u,v}^{-1}(U_1)$ and $B \in \omega_{u,v}^{-1}(U_2)$. Hence τ_w is Hausdorff, and the proof is complete. □

In particular, this guarantees uniqueness of limit for weakly convergent sequences (Proposition A.24).

In fact, the weak operator topology is induced by a metric on bounded parts of $\mathcal{B}(\mathcal{H})$.

Theorem 1.51. The weak operator topology is metrisable on bounded subsets of $\mathcal{B}(\mathcal{H})$.

Proof. Here we only indicate the metric to consider, we check it is a metric, and we provide additional references to complete the details.

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and define a map $\mathrm{d}\colon \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \longrightarrow [0, \infty)$ by

$$\mathrm{d}(A, B) := \sum_{n,m\in\mathbb{N}} \frac{1}{2^{n+m}} |\langle (A - B)e_m, e_n\rangle|$$

for any $A, B \in \mathcal{B}(\mathcal{H})$. The first step to prove is that d is a well-defined metric on $\mathcal{B}(\mathcal{H})$.

First of all, note that if $A, B \in \mathcal{B}(\mathcal{H})$, the Cauchy-Schwarz inequality provides

$$\frac{1}{2^{n+m}}|\langle (A-B)e_m, e_n\rangle| \leq \frac{1}{2^{n+m}}\|A-B\|\|e_m\|\|e_n\| = \frac{1}{2^{n+m}}\|A-B\|$$

and $\sum_{n,m\in\mathbb{N}} \frac{1}{2^{n+m}} < \infty$, whence $\mathrm{d}(A,B)$ is finite, and positive.

Also, $\mathrm{d}(A,A) = 0$ for any $A \in \mathcal{B}(\mathcal{H})$. Conversely, if $\mathrm{d}(A,B) = 0$ for some $A, B \in \mathcal{B}(\mathcal{H})$, then

$$\langle (A-B)e_m, e_n\rangle = 0$$

for all $n, m \in \mathbb{N}$. Thus A and B agree[10] on each e_n, $n \in \mathbb{N}$, and by linearity they agree on $\mathrm{Vect}((e_n)_{n\in\mathbb{N}})$. Since this subspace is dense in $\mathcal{H}$ (as $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis), A and B must agree on $\mathcal{H}$, so $A = B$.

The symmetry of d is obvious from the definition, and if $A, B, C \in \mathcal{B}(\mathcal{H})$, then

$$\begin{aligned}
\mathrm{d}(A,C) &= \sum_{n,m\in\mathbb{N}} \frac{1}{2^{n+m}}|\langle (A-C)e_m, e_n\rangle| \\
&\leq \sum_{n,m\in\mathbb{N}} \frac{1}{2^{n+m}}\left(|\langle (A-B)e_m, e_n\rangle| + |\langle (B-C)e_m, e_n\rangle|\right) \\
&= \sum_{n,m\in\mathbb{N}} \frac{1}{2^{n+m}}|\langle (A-B)e_m, e_n\rangle| + \sum_{n,m\in\mathbb{N}} \frac{1}{2^{n+m}}|\langle (B-C)e_m, e_n\rangle| \\
&= \mathrm{d}(A,B) + \mathrm{d}(B,C)
\end{aligned}$$

which shows the triangle inequality. Thus d is a metric on $\mathcal{B}(\mathcal{H})$.

For the rest of the proof, we refer to [6, proposition 9.1.3]. □

On the other hand, the next result guarantees we have sufficiently reduced the number of open sets to attain compactness.

Theorem 1.52. Let $A \in \mathcal{B}(\mathcal{H})$. The closed norm ball

$$B'_{\|\cdot\|}(A,\varepsilon) = \{B \in \mathcal{B}(\mathcal{H}) : \|A-B\| \leq \varepsilon\}$$

is weak operator compact.

Proof. Here also we only sketch the main idea of the proof, and we provide references for the missing details.

---

(10) Indeed, the fact that $\langle (A-B)e_m, e_n\rangle = 0$ for all $n, m \in \mathbb{N}$ precisely means that, for a fixed $m \in \mathbb{N}$, $(A-B)e_m$ is orthogonal to e_n, and thus by linearity to $\mathrm{Vect}((e_n)_{n\in\mathbb{N}})$. Since this space is dense in $\mathcal{H}$, its orthogonal is reduced to 0, so $(A-B)e_m = 0$ for any fixed $m \in \mathbb{N}$.

It is enough to prove the theorem for $A = 0$. Fix $\varepsilon > 0$, and for $u, v \in \mathcal{H}$, consider

$$D_{u,v} := \{z \in \mathbb{C} : |z| \leq \varepsilon \|u\| \|v\|\}$$

which is a compact subset of $\mathbb{C}$. Note that $\langle Bu, v\rangle \in D_{u,v}$ if $B \in B'_{\|\cdot\|}(0, \varepsilon)$. We can then define

$$\begin{aligned} \alpha \colon B'_{\|\cdot\|}(0, \varepsilon) &\longrightarrow \prod_{u,v \in \mathcal{H}} D_{u,v} \\ B &\longmapsto (\langle Bu, v\rangle)_{u,v\in\mathcal{H}}. \end{aligned}$$

The map α is injective, because if $\alpha(B_1) = \alpha(B_2)$, then $\langle B_1 u, v\rangle = \langle B_2 u, v\rangle$ for all $u, v \in \mathcal{H}$, whence $B_1 = B_2$ by Lemma 1.3. Next, we claim that α is continuous. To prove this, Proposition A.40 tells us it is enough to check the continuity of

$$\pi_{u,v} \circ \alpha \colon B'_{\|\cdot\|}(0, \varepsilon) \longrightarrow D_{u,v}$$

where $\pi_{u,v}$ is the natural projection. Now Theorem A.28 (that applies since the weak operator topology is metrisable on a bounded ball) ensures it is enough to check the sequential continuity of this map. Take then $(A_n)_{n\in\mathbb{N}}$ a weakly convergent sequence in $B'_{\|\cdot\|}(0, \varepsilon)$, and denote A its weak limit. This means that $\langle A_n u, v\rangle \longrightarrow \langle Au, v\rangle$ as $n \to \infty$, which says exactly that

$$(\pi_{u,v} \circ \alpha)(A_n) \longrightarrow (\pi_{u,v} \circ \alpha)(A)$$

as $n \to \infty$. Thus α is continuous. It is then a continuous bijective map onto its range. From there, one can show that its inverse is also continuous, and that its image is closed in $\prod_{u,v} D_{u,v}$. As the latter is compact by Tychonoff's theorem (see Appendix A), the image of α is compact, and Theorem A.34 ensures then that $B'_{\|\cdot\|}(0, \varepsilon)$ is compact for the weak operator topology. See [21, theorem 5.1.3] for further details. □

2. The cone $\mathcal{P}(\mathcal{H})$

This chapter is devoted to the study of the subset of positive invertible linear operators on a Hilbert space. We turn this set into a metric space, and we describe geodesics for the corresponding distance. We define an isometric action of $\mathrm{Aut}(\mathcal{H})$ on this metric space that preserves those geodesics. We establish the Löwner-Heinz inequality, and we derive numerous operator inequalities, in particular the Corach-Porta-Recht inequality and the Cordes inequality, following and completing the exposition of [18]. We show a convexity inequality for the distance between two geodesics.

2.1 A metric structure on $\mathcal{P}(\mathcal{H})$

As before, fix a complex separable Hilbert space $\mathcal{H}$.

First of all, observe that $\mathcal{P}(\mathcal{H})$ is not a vector subspace of $\mathcal{B}(\mathcal{H})$, because the zero operator is not invertible. Also it is not closed under arbitrary linear combinations or scalar multiplications: for instance $-A \notin \mathcal{P}(\mathcal{H})$ if $A \in \mathcal{P}(\mathcal{H})$. Nevertheless, it is closed for multiplication by strictly positive scalars and for the sum.

Lemma 2.1. Let $A, B \in \mathcal{P}(\mathcal{H})$, $\lambda > 0$. Then $\lambda A,\ A + B \in \mathcal{P}(\mathcal{H})$.

Proof. If $A \in \mathcal{P}(\mathcal{H})$ and $\lambda > 0$, then

$$\langle(\lambda A)u, u\rangle = \lambda\langle Au, u\rangle \geq 0$$

for any $u \in \mathcal{H}$, and λA is invertible of inverse $\frac{1}{\lambda}A^{-1}$. This shows that $\lambda A \in \mathcal{P}(\mathcal{H})$. Now let $A, B \in \mathcal{P}(\mathcal{H})$. Then

$$\langle(A + B)u, u\rangle = \langle Au, u\rangle + \langle Bu, u\rangle \geq 0$$

for any $u \in \mathcal{H}$, so $A + B$ is positive. Moreover, as A, B are invertible, Corollary 1.23 gives the existence of $\varepsilon_1, \varepsilon_2 > 0$ so that

$$\langle Au, u\rangle \geq \varepsilon_1\|u\|^2,\ \langle Bu, u\rangle \geq \varepsilon_2\|u\|^2$$

for all $u \in \mathcal{H}$. It is now enough to note that

$$\langle(A + B)u, u\rangle \geq (\varepsilon_1 + \varepsilon_2)\|u\|^2$$

for all $u \in \mathcal{H}$ to conclude that $A + B$ is invertible, again by Corollary 1.23. Henceforth, $A + B \in \mathcal{P}(\mathcal{H})$. □

Being closed for the sum and for the multiplication by strictly positive scalars, we say that $\mathcal{P}(\mathcal{H})$ is a *cone* inside $\mathcal{B}(\mathcal{H})$.

This cone can be endowed with a metric structure, defining the map

$$\begin{aligned} \mathrm{d}\colon \mathcal{P}(\mathcal{H})\times\mathcal{P}(\mathcal{H}) &\longrightarrow [0,\infty) \\ (A,B) &\longmapsto \|\ln(A^{-1/2}BA^{-1/2})\|. \end{aligned}$$

Before proving d is indeed a metric, we recall the basics from the theory of group actions.

Definition 2.2. Let X be a set, and G be a group.
An action of G on X, denoted $G \curvearrowright X$, is a map

$$\begin{aligned} G\times X &\longrightarrow X \\ (g,x) &\longmapsto g\cdot x \end{aligned}$$

so that $e_G\cdot x = x$ for any $x\in X$ and $(gh)\cdot x = g\cdot(h\cdot x)$ for any $x\in X$ and $g,h\in G$.

When a group G acts on a set X, we call X a $G-$*space*.

Equivalently, a group action is a group homomorphism $G \longrightarrow S(X)$, where $S(X)$ is the group of bijections on X, equipped with the composition of applications.

We call a group action $G \curvearrowright X$ *transitive* if for any $x,y\in X$, there is $g\in G$ so that $y = g\cdot x$, and we call it *faithful* if for every $g\in G$, $g\neq e_G$, there exists $x\in X$ so that $g\cdot x\neq x$.

For a group action $G \curvearrowright X$, we write O_x for the *orbit* of $x\in X$, defined as

$$O_x := \{y\in X : y = g\cdot x \text{ for some } g\in G\}$$

while $\mathrm{Stab}(x)$ stands for the *stabilizer* of $x\in X$:

$$\mathrm{Stab}(x) := \{g\in G : g\cdot x = x\}.$$

For $x\in X$, the map

$$\begin{aligned} \varphi_x\colon G &\longrightarrow O_x \\ g &\longmapsto g\cdot x \end{aligned}$$

is well-defined, surjective, and if $g,h\in G$ are so that $g^{-1}h\in\mathrm{Stab}(x)$, then $(g^{-1}h)\cdot x = x$, so $h\cdot x = g\cdot x$, and thus $\varphi_x(g) = \varphi_x(h)$. Therefore, φ_x passes to quotient and induces a bijection of sets

$$G/\mathrm{Stab}(x) \longrightarrow O_x.$$

Lastly, we denote X^G the set of $G-$*fixed points* in X, *i.e.*

$$X^G := \{x\in X : \forall g\in G,\ g\cdot x = x\}.$$

If (X,d_X) is a $G-$metric space, we say G acts *by isometries* on X if

$$\mathrm{d}_X(g\cdot x, g\cdot y) = \mathrm{d}_X(x,y)$$

for all $g \in G$ and all $x, y \in X$. This is the same as requiring that the associated homomorphism has image contained in $\mathrm{Isom}(X) \subset S(X)$, where the latter stands for the subgroup of $S(X)$ consisting of isometries of X.

A group action $G \curvearrowright X$ is said to be *continuous* if for any $g \in G$ the map $X \longrightarrow X$, $x \longmapsto g \cdot x$ is continuous with respect to the topology induced by d_X. Note that since $x \longmapsto g \cdot x$ is a bijection for any $g \in G$, whose inverse is $x \longmapsto g^{-1} \cdot x$, a group action $G \curvearrowright X$ is continuous if and only if $x \longmapsto g \cdot x$ is a homeomorphism, for any $g \in G$.

Let us now go back to positive invertible operators. We define an action of $\mathrm{Aut}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$ by

$$\begin{aligned} \mathrm{Aut}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) &\longrightarrow \mathcal{P}(\mathcal{H}) \\ (A, P) &\longmapsto A \cdot P := APA^*. \end{aligned}$$

If A is invertible and P is positive, then

$$\langle APA^* u, u \rangle = \langle PA^* u, A^* u \rangle \geq 0$$

for any $u \in \mathcal{H}$, so APA^* is positive. It is also an invertible operator as a product of invertible operators. The above map is thus well-defined. It is in fact a group action, since $\mathrm{Id}_{\mathcal{H}} \cdot P = \mathrm{Id}_{\mathcal{H}} P \mathrm{Id}_{\mathcal{H}}^* = P$ for any $P \in \mathcal{P}(\mathcal{H})$, and

$$(AB) \cdot P = (AB)P(AB)^* = A(BPB^*)A^* = A \cdot (B \cdot P).$$

for all $A, B \in \mathrm{Aut}(\mathcal{H})$, $P \in \mathcal{P}(\mathcal{H})$.

Here are two important properties of this action.

Lemma 2.3. The action $\mathrm{Aut}(\mathcal{H}) \curvearrowright \mathcal{P}(\mathcal{H})$ is transitive and continuous.

Proof. For the transitivity, it is enough to prove that we can go from $\mathrm{Id}_{\mathcal{H}} \in \mathcal{P}(\mathcal{H})$ to any other $P \in \mathcal{P}(\mathcal{H})$ by the action of an element of $\mathrm{Aut}(\mathcal{H})$. Fix such a $P \in \mathcal{P}(\mathcal{H})$. Let $A := \sqrt{P}$, which is invertible by Theorem 1.42 since P is invertible. Since the square root of a positive operator is self-adjoint, we get

$$A \cdot \mathrm{Id}_{\mathcal{H}} = AA^* = \sqrt{P}\sqrt{P} = P$$

establishing the transitivity. Let now $A \in \mathrm{Aut}(\mathcal{H})$. The continuity of $P \longmapsto APA^*$ follows from Lemma 1.47 and the fact that the composition of continuous maps is a continuous map (Example A.4(iv)). □

Let us then deduce another model for the set $\mathcal{P}(\mathcal{H})$.

Corollary 2.4. There is a bijection of sets

$$\mathrm{Aut}(\mathcal{H})/\mathcal{U}(\mathcal{H}) \cong \mathcal{P}(\mathcal{H}).$$

Proof. Since $\mathrm{Aut}(\mathcal{H}) \curvearrowright \mathcal{P}(\mathcal{H})$ is transitive, any two positive invertible operators are in the same orbit. We can then choose our favorite basepoint, namely $\mathrm{Id}_{\mathcal{H}}$, and as shown above there is a bijection of sets

$$\mathrm{Aut}(\mathcal{H})/\mathrm{Stab}(\mathrm{Id}_{\mathcal{H}}) \longrightarrow \mathcal{P}(\mathcal{H}).$$

Now $\mathrm{Stab}(\mathrm{Id}_{\mathcal{H}}) = \{B \in \mathrm{Aut}(\mathcal{H}) : BB^* = \mathrm{Id}_{\mathcal{H}}\}$, and since B is invertible, the condition $BB^* = \mathrm{Id}_{\mathcal{H}}$ also implies $B^*B = \mathrm{Id}_{\mathcal{H}}$, as seen right after Proposition 1.9. Henceforth, $\mathrm{Stab}(\mathrm{Id}_{\mathcal{H}}) = \mathcal{U}(\mathcal{H})$, and we are done. □

In the simple case where one of the operators is the identity $\mathrm{Id}_{\mathcal{H}}$, we can compute the distance explicitly.

Lemma 2.5. Let $A \in \mathcal{P}(\mathcal{H})$. Then

$$\mathrm{d}(\mathrm{Id}_{\mathcal{H}}, A) = \|\ln(A)\| = \max(\ln(\|A\|), \ln(\|A^{-1}\|)).$$

Proof. By Theorem 1.30(i), one has

$$\begin{aligned}
\|\ln(A)\| &= \|\ln\|_{C(\sigma(A))} \\
&= \max_{\lambda\in\sigma(A)} |\ln(\lambda)| \\
&= \max\left(\ln\left(\max_{\lambda\in\sigma(A)} \lambda\right), -\ln\left(\min_{\lambda\in\sigma(A)} \lambda\right)\right) \\
&= \max\left(\ln(\|A\|), -\ln\left(\frac{1}{\|A^{-1}\|}\right)\right) \\
&= \max(\ln(\|A\|), \ln(\|A^{-1}\|))
\end{aligned}$$

where, for the fourth equality, we used Corollary 1.25(ii) and Theorem 1.35 to get

$$\|A^{-1}\| = r(A^{-1}) = \max_{\mu\in\sigma(A^{-1})} \mu = \max_{\lambda\in\sigma(A)} \frac{1}{\lambda} = \frac{1}{\min_{\lambda\in\sigma(A)} \lambda}.$$

This concludes the proof of the lemma. □

Here is the main result of this subsection.

Proposition 2.6. The map d is a metric on $\mathcal{P}(\mathcal{H})$. Moreover, $\mathrm{Aut}(\mathcal{H})$ acts by isometries on $(\mathcal{P}(\mathcal{H}), \mathrm{d})$.

Proof. We start by proving the second claim. Let $A \in \mathrm{Aut}(\mathcal{H})$, $B \in \mathcal{P}(\mathcal{H})$. By Theorem 1.46, A has a polar decomposition $A = PU$ with P positive and $U \in \mathcal{U}(\mathcal{H})$. We then compute that

$$\mathrm{d}(A \cdot \mathrm{Id}_{\mathcal{H}}, A \cdot B) = \mathrm{d}(AA^*, ABA^*)$$

$$
\begin{aligned}
&= \mathrm{d}(PUU^*P^*, PUBU^*P) \\
&= \mathrm{d}(P^2, PUBU^*P) \\
&= \|\ln(P^{-1}PUBU^*PP^{-1})\| \\
&= \|\ln(UBU^*)\| \\
&= \|U\ln(B)U^*\| \\
&= \|\ln(B)\| \\
&= \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, B)
\end{aligned}
$$

where the third equality relies on the fact that U is unitary and P is self-adjoint. The fourth is the definition of d, the sixth is Remark 1.31, and the seventh also uses $U \in \mathcal{U}(\mathcal{H})$[11]. This computation allows us to handle the general case. Let now $B, C \in \mathcal{P}(\mathcal{H})$. Write $S := B^{1/2}$ and $T := S^{-1} \cdot C = S^{-1}CS^{-1}$, to get

$$
\begin{aligned}
\mathrm{d}(A \cdot B, A \cdot C) &= \mathrm{d}(A \cdot (S \cdot \mathrm{Id}_{\mathcal{H}}), A \cdot (S \cdot T)) \\
&= \mathrm{d}((AS) \cdot \mathrm{Id}_{\mathcal{H}}, (AS) \cdot T) \\
&= \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, T) \\
&= \mathrm{d}(S \cdot \mathrm{Id}_{\mathcal{H}}, S \cdot T) \\
&= \mathrm{d}(B, C)
\end{aligned}
$$

using that $\mathrm{Aut}(\mathcal{H}) \curvearrowright \mathcal{P}(\mathcal{H})$ is an action for the second equality, and the previous computation for the third and fourth equality. Hence the action of $\mathrm{Aut}(\mathcal{H})$ preserves the map d[12].

Now we turn to show that d is indeed a metric. The idea is to check directly properties of a metric in the special case where one of the operators is the identity, and then handle the general case using invariance of d under the action of $\mathrm{Aut}(\mathcal{H})$. For instance, if $A \in \mathcal{P}(\mathcal{H})$, it follows from Lemma 2.5 that

$$
\mathrm{d}(\mathrm{Id}_{\mathcal{H}}, A) = \|\ln(A)\| = \|\ln(A^{-1})\| = \|\ln(A^{-1/2}\mathrm{Id}_{\mathcal{H}}A^{-1/2})\| = \mathrm{d}(A, \mathrm{Id}_{\mathcal{H}})
$$

and if in addition $B \in \mathcal{P}(\mathcal{H})$, we can write

$$
\begin{aligned}
\mathrm{d}(A, B) &= \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, A^{-1/2} \cdot B) \\
&= \mathrm{d}(A^{-1/2} \cdot B, \mathrm{Id}_{\mathcal{H}}) \\
&= \mathrm{d}(A^{1/2} \cdot (A^{-1/2} \cdot B), A^{1/2} \cdot \mathrm{Id}_{\mathcal{H}}) \\
&= \mathrm{d}(B, A)
\end{aligned}
$$

which means that d is symmetric.

(11) In fact, we use here that if $U \in \mathcal{U}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$, then $\|UA\| = \|A\|$. Indeed, as U is unitary one has $\|U\| = \|U^{-1}\| = 1$ (as a direct consequence of Proposition 1.9) and by submultiplicativity it holds that $\|UA\| \le \|U\|\|A\| = \|A\|$. Additionally, $\|A\| = \|U^{-1}UA\| \le \|U^{-1}\|\|UA\| = \|UA\|$. Hence $\|UA\| = \|A\|$ as claimed.

(12) Note that strictly speaking, we cannot say it is an action by isometries at this point, because we did not prove that d is a metric yet.

For the triangle inequality, fix $A, B \in \mathcal{P}(\mathcal{H})$, and observe that

$$\begin{aligned}
\mathrm{d}(A,B) &= \|\ln(A^{-1/2}BA^{-1/2})\| \\
&= \max(\ln(\|A^{1/2}B^{-1}A^{1/2}\|), \ln(\|A^{-1/2}BA^{-1/2}\|)) \\
&\leq \max(\ln(\|A^{1/2}\|\|B^{-1}\|\|A^{1/2}\|), \ln(\|A^{-1/2}\|\|B\|\|A^{-1/2}\|)) \\
&= \max(\ln(\|A\|\|B^{-1}\|), \ln(\|A^{-1}\|\|B\|)) \\
&= \max(\ln(\|A\|) + \ln(\|B^{-1}\|), \ln(\|A^{-1}\|) + \ln(\|B\|)) \\
&\leq \max(\ln(\|A\|), \ln(\|A^{-1}\|)) + \max(\ln(\|B\|), \ln(\|B^{-1}\|)) \\
&= \|\ln(A)\| + \|\ln(B)\| \\
&= \mathrm{d}(A, \mathrm{Id}_{\mathcal{H}}) + \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, B)
\end{aligned}$$

using definition of d for the first and last equality. The second and the fifth one follows from Lemma 2.5, while the third one is an application of the formula $\|S^2\| = \|S\|^2$ for self-adjoint operators (Proposition 1.5(iv)). The first inequality follows from the submultiplicativity of the norm. This computation implies the triangle inequality for arbitrary operators $A, B, C \in \mathcal{P}(\mathcal{H})$, as

$$\begin{aligned}
\mathrm{d}(A,C) &= \mathrm{d}(B^{-1/2}\cdot A, B^{-1/2}\cdot C) \\
&\leq \mathrm{d}(B^{-1/2}\cdot A, \mathrm{Id}_{\mathcal{H}}) + \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, B^{-1/2}\cdot C) \\
&= \mathrm{d}(A,B) + \mathrm{d}(B,C)
\end{aligned}$$

using twice that d is invariant under the action of $\mathrm{Aut}(\mathcal{H})$, and the above computation to get the upper bound.

Finally, in the case of a distance equal to 0, we have

$$\begin{aligned}
\mathrm{d}(A,B) = 0 &\iff \|\ln(A^{-1/2}BA^{-1/2})\| = 0 \\
&\iff \ln(A^{-1/2}BA^{-1/2}) = 0 \\
&\iff A^{-1/2}BA^{-1/2} = \mathrm{Id}_{\mathcal{H}} \\
&\iff A = B.
\end{aligned}$$

Here the third equivalence relies on $\ln\colon \mathcal{P}(\mathcal{H}) \longrightarrow \mathcal{S}(\mathcal{H})$ being a bijection. We conclude that d is a metric on $\mathcal{P}(\mathcal{H})$, and that $\mathrm{Aut}(\mathcal{H})$ acts by isometries on $(\mathcal{P}(\mathcal{H}), \mathrm{d})$. □

2.2 Geodesics in $\mathcal{P}(\mathcal{H})$

Once we defined a distance on a set, one can ask whether two distinct points of the set can always be joined by a path compatible with this distance. Such paths are called geodesics, and this part aims at proving that $(\mathcal{P}(\mathcal{H}), \mathrm{d})$ is a geodesic metric space.

More precisely, if (X, d_X) is a metric space and $x, y \in X$, a *geodesic* between $x, y \in X$ is an isometric map $\sigma\colon I \longrightarrow X$, where $I = [a,b] \subset \mathbb{R}$ is an interval, *i.e.* a map so that

$$\mathrm{d}_X(\sigma(t), \sigma(t')) = |t' - t| \tag{3}$$

for any $t, t' \in I$, and so that $\sigma(a) = x$, $\sigma(b) = y$. We do not require σ to be surjective, whence the term "isometric map" rather than "isometry". Note that preserving the distance forces to have $|b - a| = \mathrm{d}_X(x, y)$, and up to pre-composing with a translation, we may assume $a = 0$ and $b = \mathrm{d}_X(x, y)$. Lastly, it will be convenient to rescale $I = [0, \mathrm{d}_X(x, y)]$ to the unit interval $[0, 1]$, which turns condition (3) above in

$$\mathrm{d}_X(\sigma(t), \sigma(t')) = \mathrm{d}_X(x, y)|t' - t|, \ \forall t, t' \in [0, 1].$$

To sum up, a geodesic between $x, y \in X$ is an isometric map between the unit interval equipped with the euclidean distance multiplied by $\mathrm{d}_X(x, y)$ and the metric space (X, d_X).

We say that X is a *geodesic* metric space if there exists a geodesic between x and y for any pair of points $x, y \in X$. Moreover, X is *uniquely geodesic* if there is a unique geodesic between x and y, for any $x, y \in X$.

Example 2.7. A normed space $(X, \|\cdot\|)$ is a metric space for the natural distance $\mathrm{d}_X(x, y) := \|x - y\|$, $x, y \in X$. It is furthermore a geodesic metric space, as for any $x, y \in X$ the map

$$\begin{aligned} \sigma \colon [0, 1] &\longrightarrow X \\ t &\longmapsto (1 - t)x + ty \end{aligned}$$

is a geodesic between x and y, because $\sigma(0) = x$, $\sigma(1) = y$, and

$$\begin{aligned} \mathrm{d}_X(\sigma(t), \sigma(t')) &= \|\sigma(t') - \sigma(t)\| \\ &= \|(1 - t')x + t'y - (1 - t)x - ty\| \\ &= \|(t - t')(x - y)\| \\ &= |t' - t| \mathrm{d}_X(x, y) \end{aligned}$$

for all $t, t' \in [0, 1]$. Additionally, X is uniquely geodesic if and only if its unit ball is *strictly convex*, in the sense that $\|(1 - t)y_1 + ty_2\| < 1$ for all distinct unit vectors $y_1, y_2 \in X$ and $t \in (0, 1)$. To see this, note that if $[x, y]$ denotes the geodesic segment between x and y we have just built, then X is uniquely geodesic if and only if for any $x, x', y \in X$, $x' \notin [x, y] \implies \mathrm{d}_X(x, y) < \mathrm{d}_X(x, x') + \mathrm{d}_X(x', y)$. Letting $x_1 := x' - x$, $x_2 := y - x'$, the last condition is equivalent to $\|x_1 + x_2\| < \|x_1\| + \|x_2\|$ whenever x_1, x_2 are linearly independent. Let then x_1, $x_2 \in X$ be linearly independent, and observe that

$$\begin{aligned} x_1 + x_2 &= (\|x_1\| + \|x_2\|)\left(\frac{x_1}{\|x_1\| + \|x_2\|} + \frac{x_2}{\|x_1\| + \|x_2\|}\right) \\ &= (\|x_1\| + \|x_2\|)\left(\frac{\|x_1\|}{\|x_1\| + \|x_2\|}\frac{x_1}{\|x_1\|} + \frac{\|x_2\|}{\|x_1\| + \|x_2\|}\frac{x_2}{\|x_2\|}\right) \\ &= (\|x_1\| + \|x_2\|)((1 - t)y_1 + ty_2) \end{aligned}$$

where $t := \frac{\|x_2\|}{\|x_1\| + \|x_2\|} \in (0, 1)$ and $y_1 := \frac{x_1}{\|x_1\|}$, $y_2 := \frac{x_2}{\|x_2\|}$ are unit vectors. Thus $\|x_1 + x_2\| < \|x_1\| + \|x_2\|$ if and only if $\|(1 - t)y_1 + ty_2\| < 1$ for every $t \in (0, 1)$ and unit vectors $y_1, y_2 \in X$, *i.e.* the unit ball of X is strictly convex.

From this result, it follows that $\mathbb{R}^n$ equipped with the euclidean norm $\|\cdot\|_2$ (and thus the associated euclidean distance d_2) is uniquely geodesic. However, if we choose rather the norm $\|\cdot\|_1$ or $\|\cdot\|_\infty$, defined as

$$\|(x_1,\ldots,x_n)\|_1 := \sum_{k=1}^{n} |x_k|, \ \|(x_1,\ldots,x_n)\|_\infty := \max(|x_1|,\ldots,|x_n|)$$

for $(x_1,\ldots,x_n) \in \mathbb{R}^n$, the metric spaces $(\mathbb{R}^n, \mathrm{d}_1)$, $(\mathbb{R}^n, \mathrm{d}_\infty)$ are not uniquely geodesic, as their unit balls are not strictly convex (cf. Figure 1).

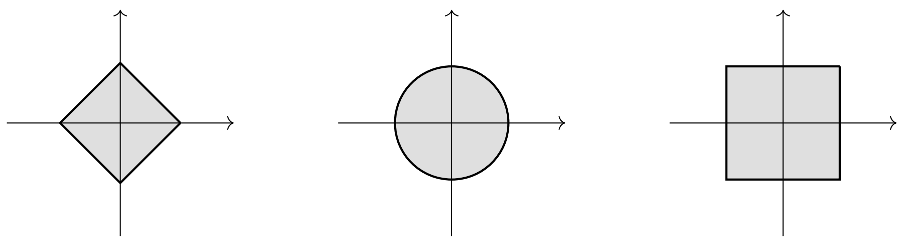

Figure 1: Unit balls in $(\mathbb{R}^2, \mathrm{d}_1)$, $(\mathbb{R}^2, \mathrm{d}_2)$ *and* $(\mathbb{R}^2, \mathrm{d}_\infty)$

For instance, in $(\mathbb{R}^2, \mathrm{d}_\infty)$, the two maps $\sigma_1, \sigma_2 \colon [0,1] \longrightarrow \mathbb{R}^2$ defined as

$$\sigma_1(t) := (t,0), \ \sigma_2(t) := \begin{cases} (t,t) & \text{if } 0 \le t \le \frac{1}{2} \\ (t,1-t) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

are both geodesics between $(0,0)$ and $(1,0)$.

For us, the most important example of geodesic metric space will be $(\mathcal{P}(\mathcal{H}), \mathrm{d})$.

Lemma 2.8. Let $A, B \in \mathcal{P}(\mathcal{H})$. The map

$$\begin{aligned} \sigma(A,B,\cdot) \colon [0,1] &\longrightarrow \mathcal{P}(\mathcal{H}) \\ t &\longmapsto A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^t A^{1/2} \end{aligned}$$

is a geodesic between A and B. Moreover, the action of $\mathrm{Aut}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$ preserves those geodesics, *i.e.*

$$\sigma(A\cdot B, A\cdot C, t) = A\cdot\sigma(B,C,t)$$

for any $A \in \mathrm{Aut}(\mathcal{H})$, $B, C \in \mathcal{P}(\mathcal{H})$ and $t \in [0,1]$.

Proof. First of all, note that $A^{-1/2}BA^{-1/2} = A^{-1/2}\cdot B \in \mathcal{P}(\mathcal{H})$ if $A, B \in \mathcal{P}(\mathcal{H})$. Hence

$$\big(A^{-1/2}BA^{-1/2}\big)^t \in \mathcal{P}(\mathcal{H})$$

for any $t \in [0,1]$, by Corollary 1.36, and thus

$$A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^t A^{1/2} = A^{1/2}\cdot\big(A^{-1/2}BA^{-1/2}\big)^t \in \mathcal{P}(\mathcal{H})$$

as well. Hence $\sigma(A, B, \cdot)$ is well-defined. We also have $\sigma(A, B, 0) = A^{1/2}\mathrm{Id}_{\mathcal{H}}A^{1/2} = A$ and

$$\sigma(A, B, 1) = A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2} = B.$$

Now, if $t, t' \in [0, 1]$, we compute that

$$\begin{aligned}
\mathrm{d}\big(\sigma(A,B,t), \sigma(A,B,t')\big) &= \mathrm{d}\big(A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^{t}A^{1/2},\ A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^{t'}A^{1/2}\big) \\
&= \mathrm{d}\big(A^{1/2}\cdot\big(A^{-1/2}BA^{-1/2}\big)^{t},\ A^{1/2}\cdot\big(A^{-1/2}BA^{-1/2}\big)^{t'}\big) \\
&= \mathrm{d}\big(\big(A^{-1/2}BA^{-1/2}\big)^{t}, \big(A^{-1/2}BA^{-1/2}\big)^{t'}\big) \\
&= \big\|\ln\big(\big(A^{-1/2}BA^{-1/2}\big)^{-t/2}\big(A^{-1/2}BA^{-1/2}\big)^{t'}\big(A^{-1/2}BA^{-1/2}\big)^{-t/2}\big)\big\| \\
&= \big\|\ln\big(\big(A^{-1/2}BA^{-1/2}\big)^{t'-t}\big)\big\| \\
&= \|(t'-t)\ln(A^{-1/2}BA^{-1/2})\| \\
&= |t'-t|\mathrm{d}(A, B)
\end{aligned}$$

using the fact that $\mathrm{Aut}(\mathcal{H})$ acts by isometries on $\mathcal{P}(\mathcal{H})$ for the third equality. The fifth and the sixth relies on Remark 1.40, and the seventh uses the definition of d. This proves that $\sigma(A, B, \cdot)$ is a geodesic between A and B.

We now turn our attention to the second claim. Fix $A \in \mathrm{Aut}(\mathcal{H})$, $B \in \mathcal{P}(\mathcal{H})$. Write $A = PU$ as in Theorem 1.46, with P positive and U unitary. Noticing that $AA^* = PUU^*P^* = P^2$, we have

$$\begin{aligned}
\sigma(A\cdot\mathrm{Id}_{\mathcal{H}}, A\cdot B, t) &= \sigma(AA^*, ABA^*, t) \\
&= (AA^*)^{1/2}\big((AA^*)^{-1/2}ABA^*(AA^*)^{-1/2}\big)^{t}(AA^*)^{-1/2} \\
&= P\big(P^{-1}PUBU^*P^*P^{-1}\big)^{t}P \\
&= P\big(UBU^{-1}\big)^{t}P \\
&= PUB^{t}U^*P^* \\
&= AB^{t}A^* \\
&= A\cdot\sigma(\mathrm{Id}_{\mathcal{H}}, B, t).
\end{aligned}$$

for any $t \in [0, 1]$. Here the fifth equality is Remark 1.31, and others use constantly that P is self-adjoint and U is unitary. If now $B, C \in \mathcal{P}(\mathcal{H})$ and $A \in \mathrm{Aut}(\mathcal{H})$, we use this computation to generalize, in the same spirit as for Proposition 2.6: set $S := B^{1/2}$ and $T := S^{-1}\cdot C = S^{-1}CS^{-1}$. It follows that

$$\begin{aligned}
\sigma(A\cdot B, A\cdot C, t) &= \sigma((AS)\cdot\mathrm{Id}_{\mathcal{H}}, (AS)\cdot T, t) \\
&= (AS)\cdot\sigma(\mathrm{Id}_{\mathcal{H}}, T, t) \\
&= A\cdot(S\cdot\sigma(\mathrm{Id}_{\mathcal{H}}, T, t)) \\
&= A\cdot\sigma(S\cdot\mathrm{Id}_{\mathcal{H}}, S\cdot T, t) \\
&= A\cdot\sigma(B, C, t)
\end{aligned}$$

for any $t \in [0, 1]$, establishing the desired claim and completing the proof. □

Unless stated otherwise, in the sequel the words "the geodesic between A and B" with $A, B \in \mathcal{P}(\mathcal{H})$ always refer to the geodesic provided by Lemma 2.8.

Let us close this subsection introducing another terminology.

Definition 2.9. Let X be a geodesic metric space. For any $x, y \in X$, fix a geodesic $\sigma(x, y, \cdot)\colon [0,1] \longrightarrow X$ between x and y.
$A \subset X$ is called metrically convex if $\sigma(x, y, t) \in A$ for any $x, y \in A$ and $t \in [0,1]$.

2.3 The Löwner-Heinz inequality

Given four operators $A, B, C, D \in \mathcal{P}(\mathcal{H})$, the maps

$$\sigma(A, B, \cdot),\ \sigma(C, D, \cdot)\colon [0,1] \longrightarrow \mathcal{P}(\mathcal{H})$$

are geodesics connecting A to B and C to D respectively, and we now wonder how behaves the distance between two geodesics with respect to the distance between A and C, and B and D. In a next part, we will establish a convexity inequality for the function

$$[0,1] \longrightarrow [0,\infty),\ t \longmapsto \mathrm{d}(\sigma(A,B,t), \sigma(C,D,t)).$$

The proof will require several *operator inequalities*. In this part, we state and establish the first one of those, called the *Löwner-Heinz inequality*.

First, let us precise in which way we compare bounded operators.

Definition 2.10. Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint.
We say that A is smaller than B, or B is larger than A, if $B - A$ is positive.

If A is smaller than B, we write $A \leq B$ or $B \geq A$. With this notation, $A \in \mathcal{B}(\mathcal{H})$ is positive if $A \geq 0$.

The relation $\leq$ is a partial, but not total, order on the class of self-adjoint operators on $\mathcal{H}$. Indeed, for $A \in \mathcal{S}(\mathcal{H})$, the operator $A - A = 0$ is positive, so $A \leq A$. Also, if $A \leq B$ and $B \leq C$, the operator $C - A = (C - B) + (B - A)$ is positive because it is the sum of two positive operators (Lemma 2.1), so $A \leq C$. Finally, if $A \leq B$ and $B \leq A$, we have

$$\langle Au, u\rangle = \langle Bu, u\rangle$$

for any $u \in \mathcal{H}$, which provides $\langle (A - B)u, u\rangle = 0$ for any $u \in \mathcal{H}$. Lemma 1.3 now implies $A - B = 0$, so $A = B$.

Moreover, this relation enjoys several useful properties for computations with positive operators.

Proposition 2.11. Let $A, B \geq 0$, and $C \in \mathcal{S}(\mathcal{H})$. The following properties hold.

(i) If $A \leq B$, then $\|A\| \leq \|B\|$.

(ii) If $A \leq B$, then $A + C \leq B + C$.

(iii) If $A \leq B$ and $\lambda > 0$, then $\lambda A \leq \lambda B$.

(iv) $A \leq \mathrm{Id}_{\mathcal{H}}$ if and only if $\|A\| \leq 1$.

(v) If A is invertible, $A \leq \mathrm{Id}_{\mathcal{H}}$ if and only if $A^{-1} \geq \mathrm{Id}_{\mathcal{H}}$.

Proof. (i) By [6, theorem 2.2.13], [13, theorem 1.12] and the fact that A, B are positive, their norms can be computed as

$$\|A\| = \sup_{\|u\|=1} \langle Au, u\rangle, \ \|B\| = \sup_{\|u\|=1} \langle Bu, u\rangle.$$

By assumption, $B \geq A$ so $B = A + P$ where P is positive. For $u \in \mathcal{H}$ with $\|u\| = 1$, we then obtain $\langle Bu, u\rangle = \langle Au, u\rangle + \langle Pu, u\rangle \geq \langle Au, u\rangle$ and thus

$$\|B\| = \sup_{\|u\|=1} \langle Bu, u\rangle \geq \sup_{\|u\|=1} \langle Au, u\rangle = \|A\|.$$

(ii) Directly, $(B + C) - (A + C) = B - A$ which is positive by assumption, whence the claim.

(iii) If $A \leq B$ and $\lambda > 0$, then $B - A \geq 0$, so $\lambda(B - A) \geq 0$ by Lemma 2.1 (more precisely, by the proof of 2.1, as A, B need not to be invertible here). Expanding the left-hand side and using (ii) with $C = \lambda A$, one gets $\lambda B \geq \lambda A$.

(iv) If $A \leq \mathrm{Id}_{\mathcal{H}}$, (i) gives directly $\|A\| \leq \|\mathrm{Id}_{\mathcal{H}}\| = 1$. Conversely, assume that $\|A\| \leq 1$. We have then

$$\max_{\lambda \in \sigma(A)} \lambda = r(A) = \|A\| \leq 1$$

and as A is positive it follows that $\sigma(A) \subset [0, 1]$. Thus $\sigma(A - \mathrm{Id}_{\mathcal{H}}) \subset (-\infty, 0]$ by Lemma 1.16, whence

$$\sigma(\mathrm{Id}_{\mathcal{H}} - A) = \{-t : t \in \sigma(A - \mathrm{Id}_{\mathcal{H}})\} \subset [0, \infty).$$

As $\mathrm{Id}_{\mathcal{H}} - A$ is self-adjoint and has positive spectrum, we conclude from Corollary 1.32(iii) that $\mathrm{Id}_{\mathcal{H}} - A$ is positive. Thus $A \leq \mathrm{Id}_{\mathcal{H}}$ and point (iv) is proved.

(v) Since $X \longmapsto X^{-1}$ is an involution on $\mathcal{P}(\mathcal{H})$ and $\mathrm{Id}_{\mathcal{H}}^{-1} = \mathrm{Id}_{\mathcal{H}}$, it is enough to prove that $A \leq \mathrm{Id}_{\mathcal{H}}$ implies $A^{-1} \geq \mathrm{Id}_{\mathcal{H}}$.

If A is invertible and $A \leq \mathrm{Id}_{\mathcal{H}}$, then $\sigma(A) \subset (0, 1]$, and Theorem 1.35 ensures that

$$\sigma(A^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(A)\} \subset [1, \infty).$$

It follows from Lemma 1.16 that $\sigma(A^{-1} - \mathrm{Id}_{\mathcal{H}}) = \sigma(A^{-1}) - 1 \subset [0, \infty)$, and additionally $A^{-1} - \mathrm{Id}_{\mathcal{H}}$ is self-adjoint, as the difference of two self-adjoint operators. Hence we conclude from Corollary 1.32(iii) that $A^{-1} - \mathrm{Id}_{\mathcal{H}}$ is positive, as wished. □

One must see in this result natural analogs of the properties that hold for positive real numbers. However, some other properties may fail. For instance, it is *not* true that for $A \in \mathcal{P}(\mathcal{H})$, $A \geq \mathrm{Id}_{\mathcal{H}}$ if and only if $\|A\| \geq 1$. It is true that having $A \geq \mathrm{Id}_{\mathcal{H}}$ implies $\|A\| \geq 1$, but the converse does not hold. Consider for instance the operator

$$A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

on $\mathcal{H} = \mathbb{C}^2$. It has norm larger than 1, but is not larger than $\mathrm{Id}_{\mathbb{C}^2}$, as one of its eigenvalues is $\frac{1}{2} < 1$.

Additionally, some properties fail because operators do not necessarily commute, unlike to real numbers. The next remark provides an example.

Remark 2.12. In general, it is not true that if $A \leq B$ and $C \geq 0$, then $AC \leq BC$. Consider for instance the operators

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

on $\mathcal{H} = \mathbb{C}^2$. The operator C is positive, because it is self-adjoint and its eigenvalues are 0 and 2. Also $B - A = A \geq 0$, but $(B - A)C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not positive, since it is even not self-adjoint.

However, and we will use this below, it is true that

$$A \leq B, \ C \in \mathrm{Aut}(\mathcal{H}) \Longrightarrow CAC^* \leq CBC^*. \tag{4}$$

Indeed, $CBC^* - CAC^* = C(B - A)C^* = C \cdot (B - A)$ is the result of the action of C on the positive operator $B - A$, and we proved above that such operators are positive. In particular, if $A, B \geq 0$, $A \leq B$, and $C \in \mathrm{Aut}(\mathcal{H})$, then

$$\|CAC^*\| \leq \|CBC^*\|$$

combining (4) and Proposition 2.11(i).

The following fact will be useful in the proof of the Löwner-Heinz inequality.

Lemma 2.13. Let $A \in \mathcal{P}(\mathcal{H})$. The map

$$\begin{aligned} [0,1] &\longrightarrow \mathcal{B}(\mathcal{H}) \\ t &\longmapsto A^t \end{aligned}$$

is continuous.

Proof. If $A = \mathrm{Id}_{\mathcal{H}}$, we are looking at a constant map, which is continuous by Example A.4(ii). Suppose then $A \neq \mathrm{Id}_{\mathcal{H}}$. In particular $\|\ln(A)\| = \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, A) > 0$. Fix $\varepsilon > 0$, and let $\delta := \frac{1}{\|\ln(A)\|} \ln\left(1 + \frac{\varepsilon}{e^{\|\ln(A)\|}}\right) > 0$. Let $t, t' \in [0,1]$ be so that $|t' - t| < \delta$. We have

$$\begin{aligned}\|A^t - A^{t'}\| &= \|e^{t\ln(A)} - e^{t'\ln(A)}\| \\ &= \|e^{t\ln(A)}(\mathrm{Id}_{\mathcal{H}} - e^{(t'-t)\ln(A)})\| \\ &\leq \|e^{t\ln(A)}\|\|\mathrm{Id}_{\mathcal{H}} - e^{(t'-t)\ln(A)}\|.\end{aligned}$$

by submultiplicativity of the norm. The first factor is bounded by $e^{\|\ln(A)\|}$ as $t \leq 1$. To bound the second factor, we go through the definition of $e^{(t'-t)\ln(A)}$, as the limit of a Cauchy sequence in $\mathcal{B}(\mathcal{H})$ (cf. Remark 1.39), and we obtain

$$\begin{aligned}\|\mathrm{Id}_{\mathcal{H}} - e^{(t'-t)\ln(A)}\| &= \left\| -\sum_{k=1}^{\infty} \frac{((t'-t)\ln(A))^k}{k!} \right\| \\ &\leq \sum_{k=1}^{\infty} \frac{|t'-t|^k \|\ln(A)\|^k}{k!} \\ &= e^{|t'-t|\|\ln(A)\|} - 1.\end{aligned}$$

Hence $\|A^t - A^{t'}\| \leq e^{\|\ln(A)\|}(e^{|t'-t|\|\ln(A)\|} - 1)$ and the latter is strictly smaller than ε since $|t' - t| < \delta$. This concludes the proof. □

Another useful fact will be the density of the dyadic rational numbers into the real numbers. We denote the set of dyadic rational numbers by $\mathbb{Z}[\frac{1}{2}]$, and explicitly

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{\frac{a}{2^b} : a \in \mathbb{Z}, b \in \mathbb{N}\right\}.$$

They form a dense subset of $\mathbb{R}$, as for any $x \in \mathbb{R}$, the sequence $\left(\frac{\lfloor 2^n x \rfloor}{2^n}\right)_{n\in\mathbb{N}}$ lies in $\mathbb{Z}[\frac{1}{2}]$ and converges to $x \in \mathbb{R}$.

We are now ready to establish the Löwner-Heinz inequality.

Theorem 2.14. Let $A, B \geq 0$. If $A \geq B$ then $A^t \geq B^t$ for any $t \in [0,1]$.

Proof. To start, suppose that A and B are invertible, and that the inequality holds for some $\alpha, \beta \in [0,1]$. We show it holds for $\frac{\alpha+\beta}{2}$ as well. By hypothesis, $B^\alpha \leq A^\alpha$, so $A^{-\alpha/2}B^\alpha A^{-\alpha/2} \leq \mathrm{Id}_{\mathcal{H}}$ by Remark 2.12. It follows that $\|A^{-\alpha/2}B^\alpha A^{-\alpha/2}\| \leq 1$ by Proposition 2.11(iv), whence

$$\|B^{\alpha/2}A^{-\alpha/2}\|^2 = \|(B^{\alpha/2}A^{-\alpha/2})^* B^{\alpha/2}A^{-\alpha/2}\| = \|A^{-\alpha/2}B^\alpha A^{-\alpha/2}\| \leq 1.$$

using the C^*−identity in $\mathcal{B}(\mathcal{H})$ for the first equality and the self-adjointness of $A^{-\alpha/2}$ for the second. In particular, $\|B^{\alpha/2}A^{-\alpha/2}\| \leq 1$. The same reasoning, using the assumption $B^\beta \leq A^\beta$, yields $\|A^{-\beta/2}B^{\beta/2}\| \leq 1$. Now,

$$A^{-(\alpha+\beta)/4}B^{(\alpha+\beta)/2}A^{-(\alpha+\beta)/4} = A^{-(\alpha+\beta)/4} \cdot B^{(\alpha+\beta)/2}$$

is positive, so we get

$$\begin{aligned}
\|A^{-(\alpha+\beta)/4}B^{(\alpha+\beta)/2}A^{-(\alpha+\beta)/4}\| &= r\big(A^{-(\alpha+\beta)/4}B^{(\alpha+\beta)/2}A^{-(\alpha+\beta)/4}\big)\\
&= r\big(A^{(\alpha-\beta)/4}A^{-(\alpha+\beta)/4}B^{(\alpha+\beta)/2}A^{-(\alpha+\beta)/4}A^{(\beta-\alpha)/4}\big)\\
&= r\big(A^{-\beta/2}B^{(\alpha+\beta)/2}A^{-\alpha/2}\big)\\
&\leq \|A^{-\beta/2}B^{(\alpha+\beta)/2}A^{-\alpha/2}\|\\
&\leq \|A^{-\beta/2}B^{\beta/2}\|\|B^{\alpha/2}A^{-\alpha/2}\|\\
&\leq 1
\end{aligned}$$

using Corollary 1.25(ii) for the first equality, Proposition 1.17 for the second one, and Proposition 1.12 for the first inequality. Using once again Proposition 2.11(iv), we see that

$$A^{-(\alpha+\beta)/4}B^{(\alpha+\beta)/2}A^{-(\alpha+\beta)/4} \leq \mathrm{Id}_{\mathcal{H}}$$

or equivalently $B^{(\alpha+\beta)/2} \leq A^{(\alpha+\beta)/2}$, as wanted.

Now $A^0 \geq B^0$, and $A^1 \geq B^1$ by assumption. Hence $A^{1/2} \geq B^{1/2}$ by what we just showed. But then also $A^{1/4} \geq B^{1/4}$, and $A^{3/4} \geq B^{3/4}$. Continuing this process shows that

$$A^d \geq B^d$$

for every $d \in \mathbb{Z}[\frac{1}{2}] \cap [0,1]$.

We explain why this implies the result for all $t \in [0,1]$. Consider $X := \{(S,T) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) : S \geq T\}$. Since the map

$$\begin{aligned}
\psi\colon \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) &\longrightarrow \mathcal{B}(\mathcal{H})\\
(A,B) &\longmapsto A - B
\end{aligned}$$

is continuous and since $\mathcal{B}(\mathcal{H})^+$ is norm-closed[13] in $\mathcal{B}(\mathcal{H})$, X is closed in $\mathcal{B}(\mathcal{H})\times\mathcal{B}(\mathcal{H})$, as $X = \psi^{-1}(\mathcal{B}(\mathcal{H})^+)$. Let furthermore

$$\begin{aligned}
\varphi_{A,B}\colon [0,1] &\longrightarrow \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})\\
t &\longmapsto (A^t, B^t).
\end{aligned}$$

[13] Indeed, if $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})^+$ converges in norm to $A \in \mathcal{B}(\mathcal{H})$, then $(A_n)_{n\in\mathbb{N}}$ converges also weakly to A, and thus

$$\langle Au, u\rangle = \lim_{n\to\infty} \langle A_n u, u\rangle$$

for any $u \in \mathcal{H}$. As $\langle A_n u, u\rangle \geq 0$ for any $u \in \mathcal{H}$ and $n \in \mathbb{N}$, we deduce that $\langle Au, u\rangle \geq 0$ for any $u \in \mathcal{H}$. Hence $A \in \mathcal{B}(\mathcal{H})^+$.

The map $\varphi_{A,B}$ is continuous by Proposition A.40, as if π_1, π_2 denote the natural projections on each factor of the product $\mathcal{B}(\mathcal{H})\times\mathcal{B}(\mathcal{H})$, the two composite $\pi_1\circ\varphi_{A,B}$, $\pi_2\circ\varphi_{A,B}$ are continuous by Lemma 2.13.

With these notations, our goal is to prove that

$$\varphi_{A,B}([0,1]) \subset X.$$

Let then $t \in [0,1]$, and choose a sequence $(d_n)_{n\in\mathbb{N}} \subset \mathbb{Z}[\frac{1}{2}] \cap [0,1]$ so that $d_n \to t$ as $n \to \infty$, which is possible by the density of $\mathbb{Z}[\frac{1}{2}] \cap [0,1]$ in $[0,1]$. Henceforth, one has

$$\varphi_{A,B}(t) = \varphi_{A,B}(\lim_{n\to\infty} d_n) = \lim_{n\to\infty} \varphi_{A,B}(d_n)$$

by continuity of $\varphi_{A,B}$. By what we proved above, $A^{d_n} \geq B^{d_n}$ for any $n \in \mathbb{N}$, *i.e.* $\varphi_{A,B}(d_n) \in X$ for any $n \in \mathbb{N}$. Hence, $\varphi_{A,B}(t)$ is the limit of a sequence of elements of X, and since X is closed in $\mathcal{B}(\mathcal{H})\times\mathcal{B}(\mathcal{H})$, we deduce $\varphi_{A,B}(t) \in X$. Thus $\varphi_{A,B}([0,1]) \subset X$, and this concludes the proof in the case where A and B are both invertible.

If now $A \geq B \geq 0$, without any assumption of invertibility, then

$$A + \varepsilon\mathrm{Id}_{\mathcal{H}} \geq B + \varepsilon\mathrm{Id}_{\mathcal{H}} \geq \varepsilon\mathrm{Id}_{\mathcal{H}}$$

for all $\varepsilon > 0$, invoking Proposition 2.11(ii). Corollary 1.23 now implies that $A + \varepsilon\mathrm{Id}_{\mathcal{H}}$, $B + \varepsilon\mathrm{Id}_{\mathcal{H}}$ are both invertible, so we may apply the first part of the proof to deduce

$$(A + \varepsilon\mathrm{Id}_{\mathcal{H}})^t \geq (B + \varepsilon\mathrm{Id}_{\mathcal{H}})^t$$

for any $t \in [0,1]$. Letting $\varepsilon \to 0$ yields $A^t \geq B^t$ for any $t \in [0,1]$ and finishes the proof. □

Remark 2.15. In general, Löwner-Heinz inequality is not true for $t > 1$. Consider for instance the two bounded operators on $\mathcal{H} = \mathbb{C}^2$ given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The operator $A - B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is positive, as seen in Remark 2.12, and B is also positive, but

$$A^2 - B^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$$

and one of the eigenvalues of this matrix is $3 - \sqrt{10} < 0$, so $A^2 \geq B^2$ does not hold.

2.4 Further operator inequalities

In this part, we derive several consequences of the Löwner-Heinz inequality.

For $A, B \in \mathcal{P}(\mathcal{H})$, we will write $A\Delta_t B$ as a shorthand for the operator

$$A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^t A^{1/2}.$$

Observe that $A\Delta_t B$ is exactly the image of $t \in [0,1]$ under the map $\sigma(A,B,\cdot)$, the geodesic we have built between A and B, in Lemma 2.8. Note also that with this notation, one has $A\Delta_t B^{-1} = (A^{-1}\Delta_t B)^{-1}$ for all $t \in [0,1]$. Indeed

$$\begin{aligned} A\Delta_t B^{-1} &= A^{1/2}\big(A^{-1/2}B^{-1}A^{-1/2}\big)^t A^{1/2} \\ &= A^{1/2}\big(A^{1/2}BA^{1/2}\big)^{-t} A^{1/2} \\ &= \big(A^{-1/2}\big(A^{1/2}BA^{1/2}\big)^t A^{-1/2}\big)^{-1} \\ &= (A^{-1}\Delta_t B)^{-1}. \end{aligned}$$

In particular, $(A\Delta_t B)^{-1} = A^{-1}\Delta_t B^{-1}$ for all $t \in [0,1]$.

We now turn to the proof of what is called the *Jensen's inequality* for operators. In this view, we will make use of the next useful lemma.

Lemma 2.16. Let $A \in \mathcal{P}(\mathcal{H})$ and $X \in \mathrm{Aut}(\mathcal{H})$. Then

$$(X^*AX)^t = X^*A^{1/2}\big(A^{1/2}XX^*A^{1/2}\big)^{t-1}A^{1/2}X$$

for all $t \in \mathbb{R}$. In particular, $A\Delta_t B = B\Delta_{1-t}A$ for any $t \in [0,1]$ and $A, B \in \mathcal{P}(\mathcal{H})$.

Proof. As X and $A^{1/2}$ are both invertible, so is their product, and we can then consider $A^{1/2}X = PU$ the polar decomposition of $A^{1/2}X$, with P positive and U unitary, according to Theorem 1.46. We have the identities

$$X^*A^{1/2} = (A^{1/2}X)^* = (PU)^* = U^*P, \;\; A^{1/2}XX^*A^{1/2} = PUU^*P = P^2$$

so for all $t \in \mathbb{R}$, we get

$$\begin{aligned} (X^*AX)^{1+t} &= (X^*A^{1/2}A^{1/2}X)^{1+t} \\ &= (U^*PPU)^{1+t} \\ &= U^*P^{2(1+t)}U \\ &= U^*PP^{2t}PU \\ &= X^*A^{1/2}\big(A^{1/2}XX^*A^{1/2}\big)^t A^{1/2}X \end{aligned}$$

using Remark 1.31 for the third equality, because $U^* = U^{-1}$. This proves the first statement, and in particular

$$(X^*)^{-1}(X^*AX)^t X^{-1} = A^{1/2}\big(A^{1/2}XX^*A^{1/2}\big)^{t-1}A^{1/2}$$

for any $A \in \mathcal{P}(\mathcal{H})$ and $X \in \mathrm{Aut}(\mathcal{H})$.

Now, let $A, B \in \mathcal{P}(\mathcal{H})$, $t \in [0,1]$. Applying the last equality with $X = A^{-1/2}$, it follows that

$$\begin{aligned} B\Delta_{1-t}A &= B^{1/2}\big(B^{-1/2}AB^{-1/2}\big)^{1-t}B^{1/2} \\ &= B^{1/2}\big(B^{1/2}A^{-1}B^{1/2}\big)^{t-1}B^{1/2} \\ &= B^{1/2}\big(B^{1/2}A^{-1/2}A^{-1/2}B^{1/2}\big)^{t-1}B^{1/2} \\ &= (A^{-1/2})^{-1}\big(A^{-1/2}BA^{-1/2}\big)^{t}(A^{-1/2})^{-1} \\ &= A\Delta_t B. \end{aligned}$$

This finishes our proof. □

An operator $X \in \mathcal{B}(\mathcal{H})$ is called a *contraction* if $\|X\| \leq 1$. Jensen's inequality states then the following.

Theorem 2.17. Let $A \in \mathcal{P}(\mathcal{H})$ and $X \in \mathrm{Aut}(\mathcal{H})$ be a contraction. Then

$$X^*A^tX \leq (X^*AX)^t$$

for any $t \in [0,1]$.

Proof. Since $\|X\| \leq 1$, $\|XX^*\| = \|X\|^2 \leq 1$ as well, thus $XX^* \leq \mathrm{Id}_{\mathcal{H}}$, by Proposition 2.11(iv), that we may apply since XX^* is positive. Moreover, as X is invertible so is XX^*, and (v) of Proposition 2.11 now gives

$$\mathrm{Id}_{\mathcal{H}} \leq (XX^*)^{-1} = (X^*)^{-1}X^{-1}.$$

Using observation (4), right after Remark 2.12, we may act on both sides of this inequality with $A^{-1/2}$ to get

$$A^{-1} = A^{-1/2}\mathrm{Id}_{\mathcal{H}}A^{-1/2} \leq A^{-1/2}(X^*)^{-1}X^{-1}A^{-1/2}$$

and by the Löwner-Heinz inequality, one has then

$$(A^{-1})^{1-t} \leq \big(A^{-1/2}(X^*)^{-1}X^{-1}A^{-1/2}\big)^{1-t} \tag{5}$$

for all $t \in [0,1]$. This implies that

$$\begin{aligned} (X^*AX)^t &= X^*A^{1/2}\big(A^{1/2}XX^*A^{1/2}\big)^{t-1}A^{1/2}X \\ &= X^*A^{1/2}\big(A^{-1/2}(X^*)^{-1}X^{-1}A^{-1/2}\big)^{1-t}A^{1/2}X \\ &\geq X^*A^{1/2}(A^{-1})^{1-t}A^{1/2}X \\ &= X^*A^tX \end{aligned}$$

using Lemma 2.16 for the first equality, and again (4) combined with (5) to get the lower bound. Thus we are done. □

This inequality has also an analog for operators that are not contractions. The price to pay is a multiplicative factor.

Corollary 2.18. Let $A \in \mathcal{P}(\mathcal{H})$ and $X \in \mathrm{Aut}(\mathcal{H})$. Then

$$X^*A^tX \le \|X\|^{2-2t}(X^*AX)^t$$

for any $t \in [0,1]$.

Proof. It suffices to apply Theorem 2.17 with $A \in \mathcal{P}(\mathcal{H})$ and $X' := \frac{X}{\|X\|}$, which is an invertible contraction. Re-arranging the obtained inequality with Proposition 2.11(iii) gives the conclusion. □

With this reformulation of Jensen's inequality in our hands, we establish two additional operator inequalities.

Lemma 2.19. Let $A, B, C, D \in \mathcal{P}(\mathcal{H})$. The following inequalities hold.

(i) $A\Delta_t B \le \|A^{1/2}C^{1/2}\|^{2-2t}(C^{-1}\Delta_t B)$ for any $t \in [0,1]$.

(ii) $C\Delta_t D \le \|B^{1/2}D^{1/2}\|^{2t}(C\Delta_t B^{-1})$ for any $t \in [0,1]$.

Proof. (i) By Corollary 2.18 with $X = A^{1/2}C^{1/2}$, we have

$$\begin{aligned} C^{1/2}A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^tA^{1/2}C^{1/2} &\le \|A^{1/2}C^{1/2}\|^{2-2t}\big(C^{1/2}A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2}C^{1/2}\big)^t \\ &= \|A^{1/2}C^{1/2}\|^{2-2t}\big(C^{1/2}BC^{1/2}\big)^t \end{aligned}$$

for all $t \in [0,1]$, whence

$$\begin{aligned} A\Delta_t B &= C^{-1/2}C^{1/2}A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^tA^{1/2}C^{1/2}C^{-1/2} \\ &\le \|A^{1/2}C^{1/2}\|^{2-2t}C^{-1/2}\big(C^{1/2}BC^{1/2}\big)^tC^{-1/2} \\ &= \|A^{1/2}C^{1/2}\|^{2-2t}(C^{-1}\Delta_t B) \end{aligned}$$

for any $t \in [0,1]$, using (4) once again to get the inequality. This establishes (i).

(ii) For all $t \in [0,1]$, one computes that

$$\begin{aligned} C\Delta_t D &= D\Delta_{1-t}C \\ &\le \|B^{1/2}D^{1/2}\|^{2-2(1-t)}(B^{-1}\Delta_{1-t}C) \\ &= \|B^{1/2}D^{1/2}\|^{2t}(C\Delta_t B^{-1}) \end{aligned}$$

using the second part of Lemma 2.16 for the first and third equality, and point (i) of the present lemma for the second inequality. The proof is complete. □

Here is the central operator inequality we will need to prove convexity of the distance between geodesics.

Theorem 2.20. Let $A, B, C, D \in \mathcal{P}(\mathcal{H})$. Then

$$\left\|(A\Delta_t B)^{1/2}(C\Delta_t D)^{1/2}\right\| \leq \left\|A^{1/2}C^{1/2}\right\|^{1-t}\left\|B^{1/2}D^{1/2}\right\|^{t}$$

for all $t \in [0,1]$.

This result is usually known as the *Corach-Porta-Recht inequality*, from the authors' names of [7], who have studied in details the space $\mathcal{P}(\mathcal{H})$ with tools from differential geometry in a series of papers in the 90's [7, 8, 9].

Proof. Fix $t \in [0,1]$, and $A, B, C, D \in \mathcal{P}(\mathcal{H})$. One computes that

$$\begin{aligned}
&\left\|(A\Delta_t B)^{1/2}(C\Delta_t D)^{1/2}\right\|^2 \\
&= \left\|(C\Delta_t D)^{1/2}(A\Delta_t B)(C\Delta_t D)^{1/2}\right\| \\
&\leq \left\|A^{1/2}C^{1/2}\right\|^{2-2t}\left\|(C\Delta_t D)^{1/2}(C^{-1}\Delta_t B)(C\Delta_t D)^{1/2}\right\| \\
&= \left\|A^{1/2}C^{1/2}\right\|^{2-2t}\left\|(C^{-1}\Delta_t B)^{1/2}(C\Delta_t D)(C^{-1}\Delta_t B)^{1/2}\right\| \\
&\leq \left\|A^{1/2}C^{1/2}\right\|^{2-2t}\left\|B^{1/2}D^{1/2}\right\|^{2t}\left\|(C^{-1}\Delta_t B)^{1/2}(C\Delta_t B^{-1})(C^{-1}\Delta_t B)^{1/2}\right\| \\
&= \left\|A^{1/2}C^{1/2}\right\|^{2-2t}\left\|B^{1/2}D^{1/2}\right\|^{2t}\left\|(C^{-1}\Delta_t B)^{1/2}(C^{-1}\Delta_t B)^{-1}(C^{-1}\Delta_t B)^{1/2}\right\| \\
&= \left\|A^{1/2}C^{1/2}\right\|^{2-2t}\left\|B^{1/2}D^{1/2}\right\|^{2t}.
\end{aligned}$$

Above, the first equality is the C^*−identity (Proposition 1.5(iv)). The two inequalities follow from Lemma 2.19, coupled with the fact that $\|CAC^*\| \leq \|CBC^*\|$ if $A \leq B$ and $C \in \mathrm{Aut}(\mathcal{H})$ (as observed right after Remark 2.12 as well). The third equality is a consequence of $C\Delta_t B^{-1} = (C^{-1}\Delta_t B)^{-1}$, as noted above. It remains to justify the *second* equality. This is done with Corollary 1.25(ii), that we may apply since

$$(C\Delta_t D)^{1/2}(C^{-1}\Delta_t B)(C\Delta_t D)^{1/2} \text{ and } (C^{-1}\Delta_t B)^{1/2}(C\Delta_t D)(C^{-1}\Delta_t B)^{1/2}$$

are positive, and Proposition 1.17:

$$\begin{aligned}
\left\|(C\Delta_t D)^{1/2}(C^{-1}\Delta_t B)(C\Delta_t D)^{1/2}\right\| &= r\left((C\Delta_t D)^{1/2}(C^{-1}\Delta_t B)(C\Delta_t D)^{1/2}\right) \\
&= r\left((C\Delta_t D)^{1/2}(C^{-1}\Delta_t B)^{1/2}(C^{-1}\Delta_t B)^{1/2}(C\Delta_t D)^{1/2}\right) \\
&= r\left((C^{-1}\Delta_t B)^{1/2}(C\Delta_t D)(C^{-1}\Delta_t B)^{1/2}\right) \\
&= \left\|(C^{-1}\Delta_t B)^{1/2}(C\Delta_t D)(C^{-1}\Delta_t B)^{1/2}\right\|.
\end{aligned}$$

Henceforth, taking the square root in the estimate

$$\left\|(A\Delta_t B)^{1/2}(C\Delta_t D)^{1/2}\right\|^2 \leq \left\|A^{1/2}C^{1/2}\right\|^{2-2t}\left\|B^{1/2}D^{1/2}\right\|^{2t}$$

yields the announced inequality. This finishes our proof. □

We deduce from this theorem the *Cordes inequality*, from the author's name of [10].

Corollary 2.21. Let $B, D \in \mathcal{P}(\mathcal{H})$. Then $\|B^t D^t\| \leq \|BD\|^t$ for all $t \in [0,1]$.

Proof. Fix $B, D \in \mathcal{P}(\mathcal{H})$ and apply Theorem 2.20 with $A = C = \mathrm{Id}_{\mathcal{H}}$ and B^2, D^2 to get the conclusion. □

Thus, we showed that the Cordes inequality is a consequence of the Löwner-Heinz inequality. It turns out the two inequalities are equivalent, and much more: they are both equivalent to Jensen's inequality, to Corach-Recht-Porta inequality, or to others *a priori* weaker inequalities. The proof of Theorem 2.14 for instance shows that $A^t \geq B^t$ for all $t \in [0,1]$ if and only if $A^{1/2} \geq B^{1/2}$, when $A \geq B \geq 0$. See [10, 18, 19] for more background on these results.

2.5 Convexity of the distance between geodesics

The next proposition is already proving the convexity inequality in a particular case. We will explain below how this particular case allows to handle the general one.

Proposition 2.22. Let $A, B \in \mathcal{P}(\mathcal{H})$. Then

$$\mathrm{d}(A^t, B^t) \leq t\,\mathrm{d}(A, B)$$

for all $t \in [0,1]$.

Proof. Let $A, B \in \mathcal{P}(\mathcal{H})$. We distinguish two cases:

(i) $\|A^{1/2}B^{-1}A^{1/2}\| \leq \|A^{-1/2}BA^{-1/2}\|$,

(ii) $\|A^{-1/2}BA^{-1/2}\| \leq \|A^{1/2}B^{-1}A^{1/2}\|$.

(i) Let us first assume that $\|A^{1/2}B^{-1}A^{1/2}\| \leq \|A^{-1/2}BA^{-1/2}\|$, and fix $t \in [0,1]$. As $\ln\colon (0,\infty) \longrightarrow \mathbb{R}$ is increasing and $t \geq 0$, we have also

$$t\ln(\|A^{1/2}B^{-1}A^{1/2}\|) \leq t\ln(\|A^{-1/2}BA^{-1/2}\|). \tag{6}$$

Now, observe that

$$\begin{aligned}
\ln(\|A^{-t/2}B^tA^{-t/2}\|) &= \ln(\|A^{-t/2}B^{t/2}B^{t/2}A^{-t/2}\|)\\
&= \ln\left(\|A^{-t/2}B^{t/2}\|^2\right)\\
&\leq \ln\left((\|A^{-1/2}B^{1/2}\|^2)^t\right)\\
&= t\ln(\|A^{-1/2}B^{1/2}\|^2)
\end{aligned}$$

$$= t\ln(\|A^{-1/2}BA^{-1/2}\|)$$

using the C^*−identity for the second and the fifth equality. The inequality follows from Corollary 2.21 and the fact that ln is increasing on $(0,\infty)$. Exactly in the same way, we have the inequality

$$\ln(\|A^{t/2}B^{-t}A^{t/2}\|) \leq t\ln(\|A^{1/2}B^{-1}A^{1/2}\|). \tag{7}$$

To conclude, we invoke Lemma 2.5 and write explicitly

$$\mathrm{d}(A^t,B^t) = \|\ln(A^{-t/2}B^tA^{-t/2})\| = \max(\ln(\|A^{-t/2}B^tA^{-t/2}\|), \ln(\|A^{t/2}B^{-t}A^{t/2}\|)).$$

Inside the maximum, the first quantity is bounded from above by $t\ln(\|A^{-1/2}BA^{-1/2}\|)$, and combining (6) and (7), so is the second quantity. Hence

$$\begin{aligned}\mathrm{d}(A^t,B^t) &= \max(\ln(\|A^{-t/2}B^tA^{-t/2}\|), \ln(\|A^{t/2}B^{-t}A^{t/2}\|))\\ &\leq t\ln(\|A^{-1/2}BA^{-1/2}\|)\\ &= t\max(\ln(\|A^{-1/2}BA^{-1/2}\|), \ln(\|A^{1/2}B^{-1}A^{1/2}\|))\\ &= t\|\ln(A^{-1/2}BA^{-1/2})\|\\ &= t\mathrm{d}(A,B).\end{aligned}$$

where the second equality follows from $\|A^{1/2}B^{-1}A^{1/2}\| \leq \|A^{-1/2}BA^{-1/2}\|$. This proves the proposition in the case (i).

(ii) Assume now that $\|A^{-1/2}BA^{-1/2}\| \leq \|A^{1/2}B^{-1}A^{1/2}\|$, and let $t \in [0,1]$. This time, we have the opposite of (6):

$$t\ln(\|A^{-1/2}BA^{-1/2}\|) \leq t\ln(\|A^{1/2}B^{-1}A^{1/2}\|).$$

Hence, when writing

$$\mathrm{d}(A^t,B^t) = \max(\ln(\|A^{-t/2}B^tA^{-t/2}\|), \ln(\|A^{t/2}B^{-t}A^{t/2}\|))$$

this is now $t\ln(\|A^{1/2}B^{-1}A^{1/2}\|)$ that bounds simultaneously from above both quantities inside the maximum, whence

$$\mathrm{d}(A^t,B^t) \leq t\ln(\|A^{1/2}B^{-1}A^{1/2}\|) = t\mathrm{d}(A,B).$$

This establishes the result in case (ii), and we are done. □

We thus see that the inequality

$$\|\ln(A^{-t/2}BA^{-t/2})\| \leq t\|\ln(A^{-1/2}BA^{-1/2})\|,\ t\in[0,1],\ A,B\in\mathcal{P}(\mathcal{H})$$

is a consequence of the Cordes inequality. As showed in [1, theorem 1], the converse holds as well, enlarging our list of equivalent operator inequalities.

We can now prove the convexity of the distance between two geodesics in $\mathcal{P}(\mathcal{H})$.

Theorem 2.23. Let $A, B, C, D \in \mathcal{P}(\mathcal{H})$. Then

$$\mathrm{d}(\sigma(A,B,t),\sigma(C,D,t)) \le (1-t)\mathrm{d}(A,C) + t\mathrm{d}(B,D)$$

for all $t \in [0,1]$.

Proof. We start by showing the result in two particular cases: (i) if $A = C$ and (ii) if $B = D$, and we will explain below why it implies the general case.

(i) Suppose that $A = C$. Since the action $\mathrm{Aut}(\mathcal{H}) \curvearrowright \mathcal{P}(\mathcal{H})$ is transitive (Lemma 2.3), preserves the metric d (Proposition 2.6) and the geodesics (Lemma 2.8), we may assume that in fact $A = C = \mathrm{Id}_{\mathcal{H}}$. In this case $\mathrm{d}(A,C) = \mathrm{d}(A,A) = 0$, and the geodesics between A and B and C and D are $\sigma(\mathrm{Id}_{\mathcal{H}}, B, t) = B^t$, $\sigma(\mathrm{Id}_{\mathcal{H}}, D, t) = D^t$ for all $t \in [0,1]$. Thus we are left to prove that

$$\mathrm{d}(B^t, D^t) \le t\mathrm{d}(B,D)$$

for all $t \in [0,1]$. This is the content of Proposition 2.22, so (i) is settled.

(ii) Now suppose that $B = D$. The inequality to show reduces to

$$\mathrm{d}(\sigma(A,B,t),\sigma(C,D,t)) \le (1-t)\mathrm{d}(A,C).$$

for all $t \in [0,1]$. By Lemma 2.16, $\sigma(A,B,t) = \sigma(B,A,1-t)$, $\sigma(C,B,t) = \sigma(B,C,1-t)$ for all $t \in [0,1]$, and these two geodesics have the same starting point, so it follows from case (i) that

$$\mathrm{d}(\sigma(A,B,t),\sigma(C,B,t)) = \mathrm{d}(\sigma(B,A,1-t),\sigma(B,C,1-t)) \le (1-t)\mathrm{d}(A,C)$$

for all $t \in [0,1]$, as desired. This proves the theorem in case (ii).

We now turn to the general case. Let $A, B, C, D \in \mathcal{P}(\mathcal{H})$ and $t \in [0,1]$. We consider $\sigma(C,B,t)$ the geodesic between C and B (see Figure 2 below), and by the triangle inequality we estimate

$$\mathrm{d}(\sigma(A,B,t),\sigma(C,D,t)) \le \mathrm{d}(\sigma(A,B,t),\sigma(C,B,t)) + \mathrm{d}(\sigma(C,B,t),\sigma(C,D,t)).$$

In the right-hand side, the first term is the distance between two geodesics that end at the same point, so we may apply case (ii) above to get

$$\mathrm{d}(\sigma(A,B,t),\sigma(C,B,t)) \le (1-t)\mathrm{d}(A,C).$$

In the same way, $\sigma(C,B,\cdot)$ and $\sigma(C,D,\cdot)$ are two geodesics starting at the same point, so by case (i) it holds

$$\mathrm{d}(\sigma(C,B,t),\sigma(C,D,t)) \le t\mathrm{d}(B,D).$$

Putting these three estimates together, we conclude that

$$\begin{aligned}\mathrm{d}(\sigma(A,B,t),\sigma(C,D,t)) &\le \mathrm{d}(\sigma(A,B,t),\sigma(C,B,t)) + \mathrm{d}(\sigma(C,B,t),\sigma(C,D,t))\\ &\le (1-t)\mathrm{d}(A,C) + t\mathrm{d}(B,D)\end{aligned}$$

as announced. This finishes the proof of the theorem. □

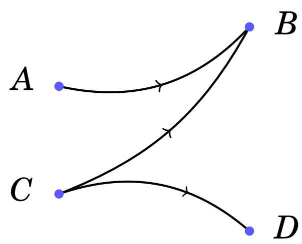

Figure 2: The idea in the proof of Theorem 2.23

Our proof of Theorem 2.23 is different from the one in [9], as it does not rely on differential geometry, and of the one in [1], as in this paper we were unable to explain one of the authors' argument.

From the convexity inequality, we derive the next corollary.

Corollary 2.24. Metric balls in $(\mathcal{P}(\mathcal{H}), \mathrm{d})$ are metrically convex.

Proof. Let $A \in \mathcal{P}(\mathcal{H})$, and $\varepsilon > 0$. Suppose that $B, C \in B_{\mathrm{d}}(A, \varepsilon)$. Let $t \in [0, 1]$. By the triangle inequality we have

$$\mathrm{d}(\sigma(B, C, t), A) \leq \mathrm{d}(\sigma(B, C, t), \sigma(B, A, t)) + \mathrm{d}(\sigma(B, A, t), A).$$

Using Theorem 2.23, we get

$$\mathrm{d}(\sigma(B, C, t), \sigma(B, A, t)) \leq (1 - t)\mathrm{d}(B, B) + t\mathrm{d}(C, A) = t\mathrm{d}(C, A) < t\varepsilon$$

for the first term, whereas

$$\mathrm{d}(\sigma(B, A, t), A) = \mathrm{d}(\sigma(B, A, t), \sigma(B, A, 1)) = (1 - t)\mathrm{d}(B, A) < (1 - t)\varepsilon$$

using that $\sigma(B, A, \cdot)$ is a geodesic and that $B \in B_{\mathrm{d}}(A, \varepsilon)$. Putting these upper bounds into the first inequality, it follows that

$$\mathrm{d}(\sigma(B, C, t), A) < t\varepsilon + (1 - t)\varepsilon = \varepsilon$$

whence $\sigma(B, C, t) \in B_{\mathrm{d}}(A, \varepsilon)$ for all $t \in [0, 1]$. This completes the proof. □

3. Unitarisability

In this part, we introduce unitarisability for groups. We show that finite and amenable groups are unitarisable, and that non-abelian free groups are not unitarisable. We investigate properties of induced actions of the group on the cone $\mathcal{P}(\mathcal{H})$. We prove Pisier's theorem, and we obtain a geometric formulation of amenability for a group.

As from the beginning, we fix $\mathcal{H}$ a complex separable Hilbert space.

3.1 The class of unitarisable groups

Let us first introduce representations of groups on Hilbert spaces.

Definition 3.1. Let G be a group.
A representation of G on $\mathcal{H}$ is a group morphism

$$\pi \colon G \longrightarrow \mathrm{Aut}(\mathcal{H}).$$

Equivalently, as observed right after Definition 2.2, a representation π of G on a Hilbert space $\mathcal{H}$ is a group action $G \curvearrowright \mathcal{H}$.

The representation π is called *unitary* if $\pi(g) \in \mathcal{U}(\mathcal{H})$ for all $g \in G$. It is called *unitarisable* if there exists $S \in \mathrm{Aut}(\mathcal{H})$ so that $S^{-1}\pi(g)S$ is unitary for all $g \in G$, and in this case S is called a *unitariser* for π. The set of unitarisers for π is denoted $U(\pi)$.

Lastly, π is said to be *uniformly bounded* if there exists a constant $C > 0$ so that $\|\pi(g)\| \leq C$, for all $g \in G$. In this case, the smallest possible bound is denoted $|\pi|$ and is called the *size* of the representation π. In fact

$$|\pi| = \sup_{g \in G} \|\pi(g)\|.$$

Example 3.2. (i) Any group G has a trivial representation, given by

$$\begin{aligned} \pi \colon G &\longrightarrow \mathrm{Aut}(\mathbb{C}) = \mathbb{C}^* \\ g &\longmapsto 1. \end{aligned}$$

It is obviously a unitary, unitarisable and uniformly bounded representation, with size $|\pi| = 1$.

(ii) If G is a subgroup of $\mathrm{GL}_n(\mathbb{C}) = \mathrm{Aut}(\mathbb{C}^n)$, $n \geq 1$, the natural injection $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is a representation of G.

(iii) Let G be a group, and $\mathcal{H} = \ell^2(G)$. The *regular* representation of G is defined as

$$\begin{aligned} \lambda_G \colon G &\longrightarrow \mathrm{Aut}(\ell^2(G)) \\ g &\longmapsto \lambda_G(g) \end{aligned}$$

where $(\lambda_G(g)(f))(x) := f(g^{-1}x)$, $f \in \ell^2(G)$, $x \in G$. This is not hard to check that λ_G is well-defined. To see it is a representation, we fix $g, h \in G$, $f \in \ell^2(G)$, and we compute

$$\begin{aligned}(\lambda_G(gh)(f))(x) &= f((gh)^{-1}x)\\ &= f(h^{-1}g^{-1}x)\\ &= (\lambda_G(h)(f))(g^{-1}x)\\ &= \lambda_G(g)(\lambda_G(h)(f))(x)\end{aligned}$$

for all $x \in G$, whence $\lambda_G(gh)(f) = \lambda_G(g)\lambda_G(h)(f)$, and this holds for all $f \in \ell^2(G)$. Thus $\lambda_G(gh) = \lambda_G(g)\lambda_G(h)$ for all $g, h \in G$, proving that λ_G is a representation of G. Furthermore, if $g \in G$ and $f \in \ell^2(G)$, one has

$$\|\lambda_G(g)(f)\|_2^2 = \sum_{x\in G} |\lambda_G(g)(f)(x)|^2 = \sum_{x\in G} |f(g^{-1}x)|^2 = \sum_{h\in G} |f(h)|^2 = \|f\|_2^2$$

whence $\|\lambda_G(g)(f)\|_2 = \|f\|_2$. Thus $\lambda_G(g)$ is an isometry for any $g \in G$. In particular, $\lambda_G(g)^*\lambda_G(g) = \mathrm{Id}_{\mathcal{H}}$ by Proposition 1.9(iv), and since $\lambda_G(g)$ is invertible, this last condition implies that $\lambda_G(g)$ is unitary, for any $g \in G$. Hence λ_G is unitary, in particular unitarisable and uniformly bounded.

(iv) If π and τ are two representations of G on $\mathcal{H}$, their *direct sum* $\pi \oplus \tau$ is the representation of G on $\mathcal{H} \oplus \mathcal{H}$ defined by

$$(\pi \oplus \tau)(g) := \begin{pmatrix}\pi(g) & 0\\ 0 & \tau(g)\end{pmatrix}, \; g \in G.$$

Concretely, for $g \in G$, $(\pi \oplus \tau)(g)$ is the linear operator on $\mathcal{H} \oplus \mathcal{H}$ defined as

$$(\pi \oplus \tau)(g)(u, v) := (\pi(g)u, \tau(g)v), \; u, v \in \mathcal{H}.$$

As $\pi(g)$ and $\tau(g)$ are both bounded, $(\pi \oplus \tau)(g)$ is also bounded[14], and since they are both invertible, so is $(\pi \oplus \tau)(g)$, hence $\pi \oplus \tau \colon G \longrightarrow \mathrm{Aut}(\mathcal{H} \oplus \mathcal{H})$ is well-defined. Lastly, as π and τ are group morphisms, we can compute

$$(\pi \oplus \tau)(gh) = \begin{pmatrix}\pi(gh) & 0\\ 0 & \tau(gh)\end{pmatrix} = \begin{pmatrix}\pi(g) & 0\\ 0 & \tau(g)\end{pmatrix}\begin{pmatrix}\pi(h) & 0\\ 0 & \tau(h)\end{pmatrix} = (\pi \oplus \tau)(g)(\pi \oplus \tau)(h)$$

for all $g, h \in G$, whence $\pi \oplus \tau$ is indeed a representation of G. More generally, if $\{\pi_n \colon G \longrightarrow \mathcal{H}_n \;:\; n \in \mathbb{N}\}$ is a family of representations of G, their direct sum is a representation of G on $\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$, usually denoted $\bigoplus_{n\in\mathbb{N}} \pi_n$, defined by

$$\Big(\bigoplus_{n\in\mathbb{N}} \pi_n\Big)(g)(u_n)_{n\in\mathbb{N}} := (\pi_n(g)u_n)_{n\in\mathbb{N}}$$

for all $g \in G$ and $(u_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$.

[14] as proved in the beginning of Chapter 1.

One has probably recognized in the regular representation of a group G the group homomorphism corresponding to the natural action of G on $\ell^2(G)$, which is used to define amenability via the Reiter property (R_2) ([13, section 2.1], [27]). Thus, as discussed in Appendix B and in [13], a group G is amenable if and only if its regular representation λ_G has $(S, \varepsilon)-$invariant vectors, for any $S \subset G$ finite and $\varepsilon > 0$.

Here goes a first connection between unitarisability and uniform boundedness.

Proposition 3.3. Let G be a group, and let π be a representation of G. If π is unitarisable, then π is uniformly bounded.

Proof. By assumption, we find $S \in \mathrm{Aut}(\mathcal{H})$ so that $\tau(g) := S^{-1}\pi(g)S$ is a unitary operator on $\mathcal{H}$, for all $g \in G$. Setting $C := \max(\|S\|, \|S^{-1}\|)$, we get

$$\|\pi(g)u\| = \|S\tau(g)S^{-1}u\| \leq \|S\|\|\tau(g)\|\|S^{-1}\|\|u\| \leq C^2\|u\|$$

for all $u \in \mathcal{H}$. Thus $\|\pi(g)\| \leq C^2$ for any $g \in G$, and π is uniformly bounded, as wished. □

In fact this proof shows that if π is unitarisable and $S \in U(\pi)$, the quantity $\|S\|\|S^{-1}\|$ bounds $\|\pi(g)\|$ from above, for all $g \in G$. In particular, $|\pi| \leq \|S\|\|S^{-1}\|$.

Definition 3.4. For $S \in \mathrm{Aut}(\mathcal{H})$, we call the real number

$$s(S) := \|S\|\|S^{-1}\|$$

the size of the operator S.

Here are the first basic properties of the size.

Proposition 3.5. If $S \in \mathrm{Aut}(\mathcal{H})$, then

(i) $s(S) \geq 1$.

(ii) $s(\lambda S) = s(S)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

(iii) $s(S) = s(S^*)$.

(iv) $s(SS^*) = s(S)^2$ and $s(\sqrt{SS^*}) = s(S)$.

Proof. As the operator norm is submultiplicative, we have $1 = \|\mathrm{Id}_{\mathcal{H}}\| = \|SS^{-1}\| \leq \|S\|\|S^{-1}\| = s(S)$. Point (ii) follows from the computation

$$s(\lambda S) = \|\lambda S\|\|(\lambda S)^{-1}\| = |\lambda|\|S\||\lambda|^{-1}\|S^{-1}\| = s(S)$$

valid for any $\lambda \in \mathbb{C} \setminus \{0\}$. Point (iii) is immediate as $\|S^*\| = \|S\|$ and $(S^*)^{-1} = (S^{-1})^*$ (Proposition 1.5, Lemma 1.6), and for (iv) we compute

$$s(S)^2 = \|S\|^2\|S^{-1}\|^2 = \|SS^*\|\|(S^{-1})^*S^{-1}\| = \|SS^*\|\|(SS^*)^{-1}\| = s(SS^*)$$

using the C^*–identity of $\mathcal{B}(\mathcal{H})$. Likewise we have

$$\begin{aligned} s(\sqrt{SS^*}) &= \|\sqrt{SS^*}\|\|(\sqrt{SS^*})^{-1}\| \\ &= \|SS^*\|^{1/2}\|(SS^*)^{-1}\|^{1/2} \\ &= \|S\|\|(S^{-1})^*S^{-1}\|^{1/2} \\ &= \|S\|\|S^{-1}\| \\ &= s(S) \end{aligned}$$

using Corollary 1.44(iii) for the second equality, and again Proposition 1.5 for the third and fourth equalities. This finishes the proof. □

In a similar manner, for π a unitarisable representation of G, the set $U(\pi)$ is also invariant under scaling. Indeed if $S \in U(\pi)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then

$$(\lambda S)^{-1}\pi(g)\lambda S = S^{-1}\pi(g)S$$

is unitary for all $g \in G$ since S unitarises π. Thus $\lambda S \in U(\pi)$.

The converse of Proposition 3.3 is not necessarily true. This motivates the next terminology.

Definition 3.6. A group G is unitarisable if all its uniformly bounded representations are unitarisable.

An inner product $(\cdot,\cdot)_{\mathcal{H}}$ on a Hilbert space $\mathcal{H}$ is called G*–invariant* if

$$(\pi(g)u, \pi(g)v)_{\mathcal{H}} = (u, v)_{\mathcal{H}}$$

for any $u, v \in \mathcal{H}$ and $g \in G$.

Note that the initial inner product $\langle\cdot,\cdot\rangle$ is G–invariant if and only if $\pi(g)$ is an isometry for any $g \in G$, and moreover since $\pi(g)$ is invertible, this is equivalent to say that $\pi(g)$ is unitary for all $g \in G$, *i.e.* π is unitary. Hence, as unitarisable representations are equivalent to unitary representations, they should give rise to inner products that are "equivalent" to $\langle\cdot,\cdot\rangle$. The next result makes this idea precise. Before stating and proving it, let us outline an idea of general interest.

Remark 3.7. If X and Y are isomorphic $\mathbb{C}$–vector spaces, via an isomorphism $f : X \longrightarrow Y$, and that X carries a hermitian inner product $\langle\cdot,\cdot\rangle_X$, we can define a hermitian inner product $\langle\cdot,\cdot\rangle_Y$ on Y, by the formula

$$\langle y_1, y_2\rangle_Y := \langle f^{-1}(y_1), f^{-1}(y_2)\rangle_X, \ y_1, y_2 \in Y.$$

Indeed, the linearity of f^{-1} and of $\langle\cdot,\cdot\rangle_X$ in the first variable immediately implies that $\langle\cdot,\cdot\rangle_Y$ is linear in the first variable, and additionally

$$\begin{aligned}\langle y_2, y_1\rangle_Y &= \langle f^{-1}(y_2), f^{-1}(y_1)\rangle_X \\ &= \overline{\langle f^{-1}(y_1), f^{-1}(y_2)\rangle_X} \\ &= \overline{\langle y_1, y_2\rangle_Y}\end{aligned}$$

for any $y_1, y_2 \in Y$. Also $\langle y,y\rangle_Y = \langle f^{-1}(y), f^{-1}(y)\rangle_X \geq 0$ for all $y \in Y$, and $\langle y,y\rangle_Y = 0$ if and only if $f^{-1}(y) = 0$, which happens if and only if $y = 0$, as f^{-1} is a linear injection. Thus $\langle\cdot,\cdot\rangle_Y$ is a hermitian inner product. We will use this observation several times below.

Lemma 3.8. Let π be a representation of G on a Hilbert space $\mathcal{H}$. The following claims are equivalent.

(i) π is unitarisable.

(ii) There exists a $G-$invariant inner product on $\mathcal{H}$ inducing the same topology as $\langle\cdot,\cdot\rangle$.

Proof. (i) $\Longrightarrow$ (ii) : Suppose that π is unitarisable, and choose a unitariser $S \in \mathrm{Aut}(\mathcal{H})$. Define a new hermitian inner product $(\cdot,\cdot)_{\mathcal{H}}$ on $\mathcal{H}$ by

$$(u,v)_{\mathcal{H}} := \langle S^{-1}u, S^{-1}v\rangle,\ u,v \in \mathcal{H}.$$

Thanks to Remark 3.7, $(\cdot,\cdot)_{\mathcal{H}}$ is indeed a hermitian inner product. In what follows, we denote $\|\cdot\|_{(\cdot,\cdot)_{\mathcal{H}}}$ the induced norm, to distinguish it from $\|\cdot\|$, induced by the initial inner product $\langle\cdot,\cdot\rangle$. First of all, $(\cdot,\cdot)_{\mathcal{H}}$ is $G-$invariant. Indeed, let $g \in G$, $u,v \in \mathcal{H}$, and compute that

$$\begin{aligned}(u,v)_{\mathcal{H}} &= \langle S^{-1}u, S^{-1}v\rangle \\ &= \langle S^{-1}\pi(g)SS^{-1}u, S^{-1}\pi(g)SS^{-1}v\rangle \\ &= \langle S^{-1}\pi(g)u, S^{-1}\pi(g)v\rangle \\ &= (\pi(g)u, \pi(g)v)_{\mathcal{H}}\end{aligned}$$

where the second equality follows from the fact that $S^{-1}\pi(g)S$ is unitary for any $g \in G$. This proves that $(\cdot,\cdot)_{\mathcal{H}}$ is $G-$invariant. It remains to prove it induces the same topology as $\langle\cdot,\cdot\rangle$. Let $\varepsilon > 0$, and set $\delta := \frac{\varepsilon}{\|S^{-1}\|} > 0$. Then, for any $u \in B_{\|\cdot\|}(0,\delta)$, one has

$$\|u\|^2_{(\cdot,\cdot)_{\mathcal{H}}} = \|S^{-1}u\|^2 \leq \|S^{-1}\|^2\|u\|^2 < \|S^{-1}\|^2\delta^2 = \varepsilon^2$$

so $\|u\|_{(\cdot,\cdot)_{\mathcal{H}}} < \varepsilon$. Hence $u \in B_{\|\cdot\|_{(\cdot,\cdot)_{\mathcal{H}}}}(0,\varepsilon)$, and we have

$$B_{\|\cdot\|}(0,\delta) \subset B_{\|\cdot\|_{(\cdot,\cdot)_{\mathcal{H}}}}(0,\varepsilon).$$

The other way around, given $\varepsilon > 0$, set $\delta := \frac{\varepsilon}{\|S\|} > 0$ to get the inclusion

$$B_{\|\cdot\|_{(\cdot,\cdot)_{\mathcal{H}}}}(0, \delta) \subset B_{\|\cdot\|}(0, \varepsilon).$$

This shows that $(\cdot, \cdot)_{\mathcal{H}}$ induces the same topology as $\langle \cdot, \cdot \rangle$, and (ii) holds.

(ii) $\Longrightarrow$ (i) : Denote $(\cdot, \cdot)_{\mathcal{H}}$ a $G-$invariant inner product on $\mathcal{H}$, inducing the same topology as $\langle \cdot, \cdot \rangle$. Here again, $\| \cdot \|_{(\cdot,\cdot)_{\mathcal{H}}}$ stands for the norm induced by $(\cdot, \cdot)_{\mathcal{H}}$. As $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$ are both separable, we can choose $E := \{e_i : i \in I\}$ and $F := \{f_j : j \in J\}$ orthonormal basis for $\langle \cdot, \cdot \rangle$ and for $(\cdot, \cdot)_{\mathcal{H}}$ respectively. These two bases must have the same cardinality (it is clear if $\mathcal{H}$ is finite-dimensional, and if $\mathcal{H}$ is infinite-dimensional we refer to [6, theorem 1.4.16]), and thus we fix a bijection $\varphi \colon I \longrightarrow J$. We use it to define a map

$$S \colon \mathrm{Vect}(E) \longrightarrow (\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$$

by $Se_i := f_{\varphi(i)}$, for any $i \in I$, and we extend S linearly to $\mathrm{Vect}(E)$. As E is an orthonormal basis for $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, this subspace is dense in $\mathcal{H}$. Moreover S is linear by construction and $\mathcal{H}$ (the target space) is complete. Additionally, if $u \in \mathrm{Vect}(E)$ is written as $u = \sum_{i=1}^{n} \lambda_i e_i$, for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, then

$$\|Su\|_{(\cdot,\cdot)_{\mathcal{H}}} = \left\| \sum_{i=1}^{n} \lambda_i f_{\varphi(i)} \right\|_{(\cdot,\cdot)_{\mathcal{H}}} = \sum_{i=1}^{n} |\lambda_i|^2 = \|u\|$$

using Pythagora's theorem ([6, proposition 1.2.2], [13, proposition 1.5]) for the last two equalities, as E and F are both orthonormal systems for $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)_{\mathcal{H}}$ respectively. This shows that $S \colon \mathrm{Vect}(E) \longrightarrow \mathcal{H}$ is an isometry, in particular it is $1-$Lipschitz and thus uniformly continuous. We can now use Proposition 1.29 to extend S in a unique way to a continuous linear map from $\mathcal{H}$ to $\mathcal{H}$, that we still denote S. In particular, S is uniformly continuous and satisfies

$$(Su, Sv)_{\mathcal{H}} = \langle u, v \rangle$$

for any $u, v \in \mathcal{H}$, and as E and F are orthonormal bases, S is a bijection. Moreover, its inverse S^{-1} also satisfies

$$\langle S^{-1}u, S^{-1}v \rangle = (u, v)_{\mathcal{H}}$$

for any $u, v \in \mathcal{H}$, and is thus uniformly continuous as well. To sum up, S is a homeomorphism from $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ to $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$. As $(\cdot, \cdot)_{\mathcal{H}}$ induces the same topology as $\langle \cdot, \cdot \rangle$, the map

$$\mathrm{Id}_{\mathcal{H}} \colon (\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}}) \longrightarrow (\mathcal{H}, \langle \cdot, \cdot \rangle).$$

is a homeomorphism as well. Thus $T := \mathrm{Id}_{\mathcal{H}} \circ S \colon (\mathcal{H}, \langle \cdot, \cdot \rangle) \longrightarrow (\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a homeomorphism, and for all $u, v \in \mathcal{H}$, $g \in G$, we have

$$(u, v)_{\mathcal{H}} = (\pi(g)u, \pi(g)v)_{\mathcal{H}} \Longleftrightarrow \langle S^{-1}u, S^{-1}v \rangle = \langle S^{-1}\pi(g)u, S^{-1}\pi(g)v \rangle$$

$$\begin{aligned}
&\iff \langle T^{-1}u, T^{-1}v\rangle = \langle T^{-1}\pi(g)u, T^{-1}\pi(g)v\rangle \\
&\iff \langle u, v\rangle = \langle T^{-1}\pi(g)Tu, T^{-1}\pi(g)Tv\rangle \\
&\iff \langle u, v\rangle = \langle (T^{-1}\pi(g)T)^{*}T^{-1}\pi(g)Tu, v\rangle.
\end{aligned}$$

Hence $\langle (T^{-1}\pi(g)T)^{*}T^{-1}\pi(g)T - \mathrm{Id}_{\mathcal{H}})u, v\rangle = 0$ for all $u, v \in \mathcal{H}$ and all $g \in G$, and Lemma 1.3 now forces $(T^{-1}\pi(g)T)^{*}T^{-1}\pi(g)T = \mathrm{Id}_{\mathcal{H}}$, $g \in G$. Coupled with the fact that $T^{-1}\pi(g)T$ is invertible for any $g \in G$, we deduce that it is in fact unitary for all $g \in G$, and thus the operator T unitarises π. This establishes (i) and concludes our proof. □

This lemma allows us to exhibit plenty examples of unitarisable groups. We begin with the case of finite groups. For the proof, note that if X is a vector space endowed with two equivalent norms $\|\cdot\|_X$, $\|\cdot\|'_X$, then the metrics d_X, d'_X corresponding to $\|\cdot\|_X$ and $\|\cdot\|'_X$ are equivalent, and thus induce the same topologies by Proposition A.9. This applies in particular to a $\mathbb{C}-$vector space X endowed with two inner products $\langle\cdot,\cdot\rangle_X$, $(\cdot,\cdot)_X$ that are equivalent, in the sense that there are constants $c, c' > 0$ so that

$$c\langle x, x\rangle_X \leq (x, x)_X \leq c'\langle x, x\rangle_X$$

for all $x \in X$.

Corollary 3.9. Finite groups are unitarisable.

Proof. Let G be a finite group, and $\pi\colon G \longrightarrow \mathrm{Aut}(\mathcal{H})$ be a uniformly bounded representation of G. Define a new inner product on $\mathcal{H}$ by setting

$$(u, v)_{\mathcal{H}} := \sum_{g\in G}\langle \pi(g)u, \pi(g)v\rangle$$

for any $u, v \in \mathcal{H}$. For all $g \in G$, the map $\mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C}$, $(u, v) \longmapsto \langle \pi(g)u, \pi(g)v\rangle$ is an inner product by Remark 3.7 (applied with $S = \pi(g^{-1})$). As a finite sum of hermitian inner products, $(\cdot,\cdot)_{\mathcal{H}}$ is indeed a hermitian inner product. It is $G-$invariant, as

$$\begin{aligned}
(\pi(h)u, \pi(h)v)_{\mathcal{H}} &= \sum_{g\in G}\langle \pi(g)\pi(h)u, \pi(g)\pi(h)v\rangle \\
&= \sum_{g\in G}\langle \pi(gh)u, \pi(gh)v\rangle \\
&= \sum_{t\in G}\langle \pi(t)u, \pi(t)v\rangle \\
&= (u, v)_{\mathcal{H}}
\end{aligned}$$

for any $h \in G$ and $u, v \in \mathcal{H}$, as π is a group morphism. Moreover, for $u \in \mathcal{H}$, one has

$$(u, u)_{\mathcal{H}} = \sum_{g\in G}\langle \pi(g)u, \pi(g)u\rangle \leq |G||\pi|^2\langle u, u\rangle$$

by Cauchy-Schwarz inequality, and on the other hand

$$\langle u, u \rangle = \langle \pi(e_G)u, \pi(e_G)u \rangle \leq (u, u)_{\mathcal{H}}.$$

Thus the topologies induced by $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)_{\mathcal{H}}$ are the same, and π is unitarisable by Lemma 3.8. We conclude that G is unitarisable. □

We now establish two stability properties for the class of unitarisable groups. The first one deals with quotient groups.

Proposition 3.10. Let G be a unitarisable group. If $N \triangleleft G$, then G/N is unitarisable.

Proof. Let $\pi \colon G/N \longrightarrow \mathrm{Aut}(\mathcal{H})$ be a uniformly bounded representation of G/N. Denote $q \colon G \longrightarrow G/N$ the quotient map, which is a surjective group homomorphism. Then $\pi \circ q$ is a representation of G on $\mathcal{H}$, and as π is uniformly bounded, so is $\pi \circ q$. As G is unitarisable, we find $S \in \mathrm{Aut}(\mathcal{H})$ so that $S^{-1}(\pi \circ q)(g)S$ is unitary for every $g \in G$. As q is surjective, $S^{-1}\pi(h)S$ is unitary for any $h \in G/N$, whence π is unitarisable. Thus G/N is unitarisable. □

The second property states that unitarisability passes to subgroups, and its proof is more involved. The idea is to see that representations of a subgroup always induce representations of the bigger group, and that the new representation is still uniformly bounded if the initial one was. Let us describe this "induction" of representation in details.

Let G be a group, H be a subgroup of G, and $\pi \colon H \longrightarrow \mathrm{Aut}(\mathcal{V})$ be a uniformly bounded representation of H on a Hilbert space $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$. Denote $C > 0$ a uniform bound for $\|\pi(h)\|$, $h \in H$. Fix also $S \subset G$ a set of representatives for the left cosets. In this way, for every $x \in G$, there is a *unique* $h_x \in H$ so that $xh_x \in S$, and we call $s_x := xh_x \in S$. Note the relations

$$h_s = e_G, \; h_{gh^{-1}} = hh_g$$

valid for every $g \in G$, $h \in H$ and $s \in S$. Indeed, if $s \in S$, h_s is the unique element of H so that $sh_s \in S$, and on the other hand $se_G = s \in S$, so $h_s = e_G$. Likewise, for $g \in G$, $h \in H$ we have

$$gh^{-1}(hh_g) = gh_g \in S$$

and the uniqueness condition provides $h_{gh^{-1}} = hh_g$.

We are going to show that the initial action of H on $\mathcal{V}$ induces an action of G on the Hilbert space $\ell^2(G/H, \mathcal{V})$. To describe this action, we first exhibit another model for this space, via an isometric isomorphism, on which we choose a natural action of the group. We then push this action through the isomorphism to obtain an action of G on $\ell^2(G/H, \mathcal{V})$.

In this view, consider the set

$$\mathcal{W} := \{\varphi \colon G \longrightarrow \mathcal{V} \mid \varphi(gh^{-1}) = \pi(h)\varphi(g) \text{ for all } g \in G,\, h \in H\}$$

with its natural structure of $\mathbb{C}-$vector space, where addition and multiplication by complex scalars are defined pointwise, using the $\mathbb{C}-$vector space structure of $\mathcal{V}$. In fact, this space is isomorphic to the $\mathbb{C}-$vector space $\mathcal{F}(G/H, \mathcal{V})$ of functions from G/H to $\mathcal{V}$. To see this, define

$$\begin{aligned} B \colon \mathcal{F}(G/H, \mathcal{V}) &\longrightarrow \mathcal{W} \\ \tilde{\varphi} &\longmapsto B\tilde{\varphi} \end{aligned}$$

where $(B\tilde{\varphi})(x) := \pi(h_x)\tilde{\varphi}(xH)$, for any $x \in G$. The other way around, set

$$\begin{aligned} \overline{B} \colon \mathcal{W} &\longrightarrow \mathcal{F}(G/H, \mathcal{V}) \\ \varphi &\longmapsto \overline{B}\varphi \end{aligned}$$

where $(\overline{B}\varphi)(xH) := \pi(h_x^{-1})\varphi(x)$, for any $xH \in G/H$.

Lemma 3.11. The maps B and $\overline{B}$ are well-defined, linear, bijective and inverses of each other.

Proof. We first prove that given $\varphi \in \mathcal{W}$, the function $\overline{B}\varphi$ is well-defined, *i.e.* its value on a left coset does not depend on a particular choice of representative for that coset. Suppose then that $xH = x'H$ for some $x, x' \in G$. This implies $s_x = s_{x'}$, so $xh_x = x'h_{x'}$, and thus $x' = xh_xh_{x'}^{-1}$. It follows that

$$\begin{aligned} (\overline{B}\varphi)(x'H) &= \pi(h_{x'}^{-1})\varphi(x') \\ &= \pi(h_{x'}^{-1})\varphi(xh_xh_{x'}^{-1}) \\ &= \pi(h_{x'}^{-1})\varphi(x(h_{x'}h_x^{-1})^{-1}) \\ &= \pi(h_{x'}^{-1})\pi(h_{x'}h_x^{-1})\varphi(x) \\ &= \pi(h_{x'}^{-1})\pi(h_{x'})\pi(h_x^{-1})\varphi(x) \\ &= \pi(h_x^{-1})\varphi(x) \\ &= (\overline{B}\varphi)(xH) \end{aligned}$$

using that $\varphi \in \mathcal{W}$ for the fourth equality, and that $\pi \colon H \longrightarrow \mathrm{Aut}(\mathcal{V})$ is a group homomorphism for the fifth one (in particular $\pi(e_G) = \mathrm{Id}_{\mathcal{V}}$). Thus $\overline{B}\varphi$ is indeed a function on the quotient G/H, so that $\overline{B}$ is well-defined.

Now, we show that B is well-defined, proving that $B\tilde{\varphi} \in \mathcal{W}$ if $\tilde{\varphi}$ is a function on G/H. Fix group elements $g \in G$, $h \in H$ and write

$$(B\tilde{\varphi})(gh^{-1}) = \pi(h_{gh^{-1}})\tilde{\varphi}(gh^{-1}H)$$

$$
\begin{aligned}
&= \pi(hh_g)\tilde{\varphi}(gH) \\
&= \pi(h)\pi(h_g)\tilde{\varphi}(gH) \\
&= \pi(h)(B\tilde{\varphi})(g)
\end{aligned}
$$

using that $h_{gh^{-1}} = hh_g$ and that π is a group homomorphism from H to $\mathrm{Aut}(\mathcal{V})$. Thus $B\tilde{\varphi} \in \mathcal{W}$ for any function $\tilde{\varphi}$ on the quotient. Additionally, if $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathcal{F}(G/H, \mathcal{V})$, $\lambda \in \mathbb{C}$ and $x \in G$, then

$$
\begin{aligned}
B(\tilde{\varphi}_1 + \lambda\tilde{\varphi}_2)(x) &= \pi(h_x)\big((\tilde{\varphi}_1 + \lambda\tilde{\varphi}_2)(xH)\big) \\
&= \pi(h_x)\big(\tilde{\varphi}_1(xH) + \lambda\tilde{\varphi}_2(xH)\big) \\
&= \pi(h_x)\tilde{\varphi}_1(xH) + \lambda\pi(h_x)\tilde{\varphi}_2(xH) \\
&= B\tilde{\varphi}_1(x) + \lambda B\tilde{\varphi}_2(x) \\
&= (B\tilde{\varphi}_1 + \lambda B\tilde{\varphi}_2)(x)
\end{aligned}
$$

whence $B(\tilde{\varphi}_1 + \lambda\tilde{\varphi}_2) = B\tilde{\varphi}_1 + \lambda B\tilde{\varphi}_2$. This shows that B is linear.

Now we prove that $B \circ \overline{B} = \mathrm{Id}_{\mathcal{W}}$. Let $\varphi \in \mathcal{W}$, and observe that

$$
\begin{aligned}
(B \circ \overline{B})(\varphi)(x) &= B(\overline{B}\varphi)(x) \\
&= \pi(h_x)(\overline{B}\varphi(xH)) \\
&= \pi(h_x)(\pi(h_x^{-1})\varphi(x)) \\
&= (\pi(h_x)\pi(h_x^{-1}))(\varphi(x)) \\
&= \mathrm{Id}_{\mathcal{V}}(\varphi(x)) \\
&= \varphi(x)
\end{aligned}
$$

for all $x \in G$. Above we used the definitions of B and $\overline{B}$, and the fact that π is a group homomorphism (in particular $\pi(e_G) = \mathrm{Id}_{\mathcal{V}}$). Thus we conclude $(B \circ \overline{B})(\varphi) = \varphi$ for every $\varphi \in \mathcal{W}$, so $B \circ \overline{B} = \mathrm{Id}_{\mathcal{W}}$. In a similar way, let $\tilde{\varphi} \in \mathcal{F}(G/H, \mathcal{V})$ and $x \in G$, and write

$$
\begin{aligned}
(\overline{B} \circ B)(\tilde{\varphi})(xH) &= \overline{B}(B\tilde{\varphi})(xH) \\
&= \pi(h_x^{-1})(B\tilde{\varphi}(x)) \\
&= \pi(h_x^{-1})(\pi(h_x)\tilde{\varphi}(xH)) \\
&= \tilde{\varphi}(xH).
\end{aligned}
$$

Thus we get $\overline{B} \circ B = \mathrm{Id}_{\mathcal{F}(G/H,\mathcal{V})}$. Consequently, $\overline{B}$ is linear as the inverse of a linear map. This terminates the proof. □

We have then an isomorphism of $\mathbb{C}-$vector spaces

$$
\mathcal{W} \cong \mathcal{F}(G/H, \mathcal{V}).
$$

Furthermore, $\mathcal{F}(G/H, \mathcal{V})$ carries the norm $\|\cdot\|_2$ defined as

$$
\|\tilde{\varphi}\|_2^2 := \sum_{s \in S} \|\tilde{\varphi}(sH)\|_{\mathcal{V}}^2, \;\; \tilde{\varphi} \in \mathcal{F}(G/H, \mathcal{V}).
$$

This norm possibly takes infinite values, and we distinguish $\ell^2(G/H,\mathcal{V})$ the complete $\mathbb{C}-$vector subspace of functions $\tilde{\varphi}$ from G/H to $\mathcal{V}$ with $\|\tilde{\varphi}\|_2^2 < \infty$. It is a Hilbert space, because its norm derives from the inner product

$$\langle\tilde{\varphi}_1,\tilde{\varphi}_2\rangle_2 := \sum_{s\in S}\langle\tilde{\varphi}_1(sH),\tilde{\varphi}_2(sH)\rangle_{\mathcal{V}}$$

and this last expression does make sense for $\tilde{\varphi}_1,\tilde{\varphi}_2 \in \ell^2(G/H,\mathcal{V})$, thanks to the Cauchy-Schwarz inequality. Now we endow $\mathcal{W}$ with a normed vector space structure, pushing the norm on $\mathcal{F}(G/H,\mathcal{V})$ through B. More precisely, for $\varphi\in\mathcal{W}$, we set

$$\|\varphi\|_{\mathcal{W}} := \|\overline{B}\varphi\|_2 = \|B^{-1}\varphi\|_2$$

and the subspace $\ell^2(G/H,\mathcal{V})$ of $\mathcal{F}(G/H,\mathcal{V})$ is identified with the subspace

$$\hat{\mathcal{W}} := \{\varphi\in\mathcal{W} : \|\varphi\|_{\mathcal{W}}^2 < \infty\}.$$

In this way, the inner product $\langle\cdot,\cdot\rangle_2$ on $\ell^2(G/H,\mathcal{V})$ is pushed along B to give rise to a hermitian inner product (by Remark 3.7) on $\hat{\mathcal{W}}$, given by

$$\langle\varphi_1,\varphi_2\rangle_{\hat{\mathcal{W}}} := \langle B^{-1}\varphi_1, B^{-1}\varphi_2\rangle_2$$

for all $\varphi_1,\varphi_2 \in \hat{\mathcal{W}}$. In particular, $(\hat{\mathcal{W}},\langle\cdot,\cdot\rangle_{\hat{\mathcal{W}}})$ is a Hilbert space, and the isomorphisms $B,\overline{B}$ are isometric.

We define an action of G on $\mathcal{W}$ as follows: for $g\in G$ and $\varphi\in\mathcal{W}$, define $g\cdot_{\mathcal{W}}\varphi$ as

$$(g\cdot_{\mathcal{W}}\varphi)(x) := \varphi(g^{-1}x)$$

for all $x\in G$. This is a group action. Indeed, if $g\in G$ and $\varphi\in\mathcal{W}$ then

$$\begin{aligned}(g\cdot_{\mathcal{W}}\varphi)(xy^{-1}) &= \varphi(g^{-1}xy^{-1})\\ &= \pi(y)\varphi(g^{-1}x)\\ &= \pi(y)(g\cdot_{\mathcal{W}}\varphi)(x)\end{aligned}$$

for any $x\in G, y\in H$, using that $\varphi\in\mathcal{W}$ for the second equality. Thus $g\cdot\varphi\in\mathcal{W}$ as well. Secondly, we have $e_G\cdot_{\mathcal{W}}\varphi=\varphi$ for any $\varphi\in\mathcal{W}$, and if $g_1,g_2\in G$ and $\varphi\in\mathcal{W}$ we compute

$$(g_1\cdot_{\mathcal{W}}(g_2\cdot_{\mathcal{W}}\varphi))(x) = (g_2\cdot_{\mathcal{W}}\varphi)(g_1^{-1}x) = \varphi(g_2^{-1}g_1^{-1}x) = \varphi((g_1g_2)^{-1}x) = ((g_1g_2)\cdot_{\mathcal{W}}\varphi)(x)$$

for any $x\in G$, which shows that $g_1\cdot_{\mathcal{W}}(g_2\cdot_{\mathcal{W}}\varphi) = (g_1g_2)\cdot_{\mathcal{W}}\varphi$.

This action of G on $\mathcal{W}$ is now transported to an action of G on $\mathcal{F}(G/H,\mathcal{V})$, setting

$$g\cdot\tilde{\varphi} := B^{-1}(g\cdot_{\mathcal{W}}B\tilde{\varphi})$$

for any $g\in G$ and $\tilde{\varphi}\in\mathcal{F}(G/H,\mathcal{V})$. We can provide an explicit formula to describe this action. Let $\tilde{\varphi}\in\mathcal{F}(G/H,\mathcal{V})$, $g\in G$, and $s\in S$, and write

$$(g\cdot\tilde{\varphi})(sH) = B^{-1}(g\cdot_{\mathcal{W}}B\tilde{\varphi})(sH)$$

$$
\begin{aligned}
&= \pi(h_s^{-1})(g \cdot_{\mathcal{W}} B\tilde{\varphi})(s) \\
&= (g \cdot_{\mathcal{W}} B\tilde{\varphi})(s) \\
&= B\tilde{\varphi}(g^{-1}s) \\
&= \pi(h_{g^{-1}s})\tilde{\varphi}(g^{-1}sH).
\end{aligned}
$$

Here the third equality follows from $h_s = e_G$ (and thus $h_s^{-1} = e_G$, so $\pi(h_s^{-1}) = \pi(e_G) = \mathrm{Id}_{\mathcal{V}}$).

The fact that π is uniformly bounded implies that this action preserves the subspace $\ell^2(G/H, \mathcal{V}) \subset \mathcal{F}(G/H, \mathcal{V})$. Indeed, if $\tilde{\varphi} \in \ell^2(G/H, \mathcal{V})$ and $g \in G$, one has

$$
\begin{aligned}
\|g \cdot \tilde{\varphi}\|_2^2 &= \sum_{s \in S} \|(g \cdot \tilde{\varphi})(sH)\|_{\mathcal{V}}^2 \\
&= \sum_{s \in S} \|\pi(h_{g^{-1}s})\tilde{\varphi}(g^{-1}sH)\|_{\mathcal{V}}^2 \\
&\leq C^2 \sum_{s \in S} \|\tilde{\varphi}(g^{-1}sH)\|_{\mathcal{V}}^2 \\
&= C^2 \|\tilde{\varphi}\|_2^2 < \infty
\end{aligned}
$$

as $\|\tilde{\varphi}\|_2^2 < \infty$.

To sum up: given a uniformly bounded representation π of H on a Hilbert space $\mathcal{V}$, we constructed an action of G on the Hilbert space $\ell^2(G/H, \mathcal{V})$, or equivalently a representation of G on $\ell^2(G/H, \mathcal{V})$. We denote it $\hat{\pi} : G \longrightarrow \mathrm{Aut}(\ell^2(G/H, \mathcal{V}))$, *i.e.*

$$
\hat{\pi}(g)\tilde{\varphi} = g \cdot \tilde{\varphi}
$$

where $g \in G$, $\tilde{\varphi} \in \ell^2(G/H, \mathcal{V})$, and we call $\hat{\pi}$ the *induced* representation of G by π.

We have now all we need to establish the following.

Proposition 3.12. Let G be a unitarisable group. If $H \leqslant G$, then H is unitarisable.

Proof. Let $\pi : H \longrightarrow \mathrm{Aut}(\mathcal{V})$ be a uniformly bounded representation of H on a Hilbert space $\mathcal{V}$. Let $C > 0$ be a uniform bound for $\|\pi(h)\|$, $h \in H$, and consider $\hat{\pi}$ the induced representation of G on $\ell^2(G/H, \mathcal{V})$. With this notation, the computation preceding this proof shows that

$$
\|\hat{\pi}(g)\tilde{\varphi}\|_2^2 \leq C^2 \|\tilde{\varphi}\|_2^2
$$

for all $g \in G$ and $\tilde{\varphi} \in \ell^2(G/H, \mathcal{V})$. Hence $\|\hat{\pi}(g)\| \leq C$ for any $g \in G$, so $\hat{\pi}$ is uniformly bounded, in fact with the same bound as π. The group G being unitarisable, $\hat{\pi}$ is unitarisable, and Lemma 3.8 provides a G−invariant inner product $[\cdot, \cdot]$ on $\ell^2(G/H, \mathcal{V})$, inducing the same topology as $\langle \cdot, \cdot \rangle_2$.

Now $\mathcal{V}$ is a subspace of $\ell^2(G/H, \mathcal{V})$, identifying a vector $v \in \mathcal{V}$ to the function $\tilde{\varphi}_v$ supported on the trivial coset $e_G H$, defined by

$$(\tilde{\varphi}_v)(sH) := \begin{cases} v & \text{if } sH = e_G H \\ 0 & \text{otherwise} \end{cases}$$

and we claim that, for $h \in H$, the action of the operator $\hat{\pi}(h)$ on this copy of $\mathcal{V}$ preserves $\mathcal{V}$ and furthermore coincides with the initial action of $\pi(h)$ on $\mathcal{V}$. For this, let $h \in H$, $v \in \mathcal{V}$. We prove that

$$\hat{\pi}(h)\tilde{\varphi}_v = \tilde{\varphi}_{\pi(h)v}$$

which shows both claims at the same time. Let $s \in S$, and write that

$$\begin{aligned}(\hat{\pi}(h)\tilde{\varphi}_v)(sH) &= \pi(h_{h^{-1}s})\tilde{\varphi}_v(h^{-1}sH) \\ &= \begin{cases} \pi(h_{h^{-1}s})v & \text{if } h^{-1}sH = e_G H \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \pi(h)v & \text{if } h^{-1}sH = e_G H \\ 0 & \text{otherwise} \end{cases} \\ &= \tilde{\varphi}_{\pi(h)v}(sH)\end{aligned}$$

because the condition $h^{-1}sH = e_G H$ is equivalent to $s \in H$, and this implies $h_{h^{-1}s} = h$. Now, observe that by definition, the restriction of $\langle \cdot, \cdot \rangle_2$ to $\mathcal{V}$ coincides with $\langle \cdot, \cdot \rangle_{\mathcal{V}}$. Thus the restriction of $[\cdot, \cdot]$ to $\mathcal{V}$ induces the same topology on $\mathcal{V}$ as $\langle \cdot, \cdot \rangle_{\mathcal{V}}$, and the representation π is H−invariant for $[\cdot, \cdot]$, as it is the restriction of a representation which is G−invariant for $[\cdot, \cdot]$. Invoking once again Lemma 3.8, π is unitarisable, and therefore so is H, as announced. The proof is complete. □

To get new examples of unitarisable groups, we establish now Dixmier's result.

Theorem 3.13. Amenable groups are unitarisable.

For the proof, we will use the characterization of amenability in terms of the existence of a G−invariant mean on $\ell^\infty(G)$ (as explained in Appendix B, theorem B.9). Note that if $m \in \mathcal{M}'(G)$ is such a G−invariant mean, then m is *increasing*, in the sense that for $f_1, f_2 \in \ell^\infty(G)$,

$$f_1 \leq f_2 \Longrightarrow m(f_1) \leq m(f_2).$$

Indeed if $f_1 \leq f_2$ then $f_2 - f_1 \geq 0$ and thus $m(f_2 - f_1) \geq 0$. The linearity of m now leads to $m(f_1) \leq m(f_2)$.

Proof. Let G be an amenable group, and denote $m \in \mathcal{M}'(G)$ a G−invariant mean. Let $\pi : G \longrightarrow \text{Aut}(\mathcal{H})$ be a uniformly bounded representation of G. For $u, v \in \mathcal{H}$, define

$$f_{u,v} : G \longrightarrow \mathbb{C}$$

$$g \longmapsto \langle \pi(g^{-1})u, \pi(g^{-1})v \rangle.$$

Let $u, v \in \mathcal{H}$. By the Cauchy-Schwarz inequality and the uniform boundedness of π, we first have

$$|f_{u,v}(g)| = |\langle \pi(g^{-1})u, \pi(g^{-1})v \rangle| \leq \|\pi(g^{-1})\|^2 \|u\|\|v\| \leq |\pi|^2 \|u\|\|v\|$$

for any $g \in G$, and thus $\|f_{u,v}\|_\infty \leq |\pi|^2 \|u\|\|v\|$. In particular, $f_{u,v} \in \ell^\infty(G)$ and we may define a map

$$\begin{aligned} (\cdot,\cdot)_{\mathcal{H}} \colon \mathcal{H} \times \mathcal{H} &\longrightarrow \mathbb{C} \\ (u, v) &\longmapsto m(f_{u,v}). \end{aligned}$$

We claim that $(\cdot,\cdot)_{\mathcal{H}}$ is a G–invariant inner product on $\mathcal{H}$. Fix $u, v, w \in \mathcal{H}$, $\lambda \in \mathbb{C}$. As $\pi(g)$ is a linear operator for all $g \in G$ and as $\langle \cdot,\cdot \rangle$ is linear in the first variable, we have

$$\begin{aligned} f_{\lambda u+v,w}(g) &= \langle \pi(g^{-1})(\lambda u + v), \pi(g^{-1})w \rangle \\ &= \lambda \langle \pi(g^{-1})u, \pi(g^{-1})w \rangle + \langle \pi(g^{-1})v, \pi(g^{-1})w \rangle \\ &= \lambda f_{u,w}(g) + f_{v,w}(g) \\ &= (\lambda f_{u,w} + f_{v,w})(g) \end{aligned}$$

for all $g \in G$, so $f_{\lambda u+v,w} = \lambda f_{u,w} + f_{v,w}$ and, as m is linear, it follows that

$$\begin{aligned} (\lambda u + v, w)_{\mathcal{H}} &= m(f_{\lambda u+v,w}) \\ &= m(\lambda f_{u,w} + f_{v,w}) \\ &= \lambda m(f_{u,w}) + m(f_{v,w}) \\ &= \lambda (u, w)_{\mathcal{H}} + (v, w)_{\mathcal{H}} \end{aligned}$$

for all $u, v, w \in \mathcal{H}$, $\lambda \in \mathbb{C}$. Hence $(\cdot,\cdot)_{\mathcal{H}}$ is linear in the first variable.

Next, by the linearity of m it holds that $m(\overline{f}) = \overline{m(f)}$ for every $f \in \ell^\infty(G)$, and thus

$$\overline{(u, v)_{\mathcal{H}}} = \overline{m(f_{u,v})} = m(\overline{f_{u,v}})$$

for all $u, v \in \mathcal{H}$. As $\langle \cdot,\cdot \rangle$ is an inner product, we also have

$$\begin{aligned} \overline{f_{u,v}}(g) &= \overline{\langle \pi(g^{-1})u, \pi(g^{-1})v \rangle} \\ &= \langle \pi(g^{-1})v, \pi(g^{-1})u \rangle \\ &= f_{v,u}(g) \end{aligned}$$

for all $g \in G$, thus $\overline{f_{u,v}} = f_{v,u}$ for all $u, v \in \mathcal{H}$. We deduce that

$$\overline{(u, v)_{\mathcal{H}}} = m(\overline{f_{u,v}}) = m(f_{v,u}) = (v, u)_{\mathcal{H}}$$

for all $u, v \in \mathcal{H}$, and $(\cdot,\cdot)_{\mathcal{H}}$ is hermitian.

Lastly, for $u \in \mathcal{H}$, one has

$$f_{u,u}(g) = \langle \pi(g^{-1})u, \pi(g^{-1})u \rangle = \|\pi(g^{-1})u\|^2 \geq 0$$

and so $f_{u,u} \geq 0$, whence $m(f_{u,u}) \geq 0$. This means that $(u,u) \geq 0$, so $(\cdot,\cdot)_{\mathcal{H}}$ is positive definite. It remains to prove it is non-degenerate, *i.e.* if $(u,u)_{\mathcal{H}} = 0$, then $u = 0$. We show rather the contrapositive. Recall that if $A \in \mathrm{Aut}(\mathcal{H})$ then $\|Au\| \geq \frac{1}{\|A^{-1}\|}\|u\|$ for all $u \in \mathcal{H}$. If $u \neq 0$ and $g \in G$, we use this observation with $A = \pi(g^{-1})$ to obtain

$$f_{u,u}(g) = \|\pi(g^{-1})u\|^2 \geq \frac{1}{|\pi|^2}\|u\|^2.$$

This holds for any $g \in G$, whence $f_{u,u} \geq \frac{1}{|\pi|^2}\|u\|^2\mathbf{1}_G$. As m is increasing, linear and takes the value 1 on $\mathbf{1}_G$, we get

$$(u,u)_{\mathcal{H}} = m(f_{u,u}) \geq m\left(\frac{1}{|\pi|^2}\|u\|^2\mathbf{1}_G\right) = \frac{1}{|\pi|^2}\|u\|^2 m(\mathbf{1}_G) = \frac{1}{|\pi|^2}\|u\|^2 > 0$$

as desired. This proves that $(\cdot,\cdot)_{\mathcal{H}}$ is a hermitian inner product.

Its $G-$invariance is a consequence of the $G-$invariance of m, as for $u,v \in \mathcal{H}$ and $g \in G$, we have

$$\begin{aligned} f_{\pi(g)u,\pi(g)v}(h) &= \langle \pi(h^{-1})\pi(g)u, \pi(h^{-1})\pi(g)v \rangle \\ &= \langle \pi(h^{-1}g)u, \pi(h^{-1}g)v \rangle \\ &= \langle \pi((g^{-1}h)^{-1})u, \pi((g^{-1}h)^{-1})v \rangle \\ &= f_{u,v}(g^{-1}h) \\ &= (gf_{u,v})(h) \end{aligned}$$

for any $h \in G$, using that π is a group homomorphism. We deduce the equality $f_{\pi(g)u,\pi(g)v} = gf_{u,v}$ for any $g \in G$ and $u,v \in \mathcal{H}$, and this provides

$$(\pi(g)u, \pi(g)v)_{\mathcal{H}} = m(f_{\pi(g)u,\pi(g)v}) = m(gf_{u,v}) = m(f_{u,v}) = (u,v)_{\mathcal{H}}$$

for all $g \in G$, $u,v \in \mathcal{H}$, whence $(\cdot,\cdot)_{\mathcal{H}}$ is $G-$invariant.

Denoting $\|\cdot\|_{(\cdot,\cdot)_{\mathcal{H}}}$ the norm induced by this new inner product, we have on the one hand

$$\|u\|_{(\cdot,\cdot)_{\mathcal{H}}} = \sqrt{(u,u)_{\mathcal{H}}} = \sqrt{m(f_{u,u})} \leq \sqrt{\|f_{u,u}\|} \leq \sqrt{|\pi|^2\|u\|^2} = |\pi|\|u\|$$

for any $u \in \mathcal{H}$. On the other hand $\|u\|^2 = \|\pi(g)\pi(g^{-1})u\|^2 \leq |\pi|^2 f_{u,u}(g)$ for any $u \in \mathcal{H}$ and $g \in G$, and as m is increasing we get

$$\|u\|^2 = m(\|u\|^2\mathbf{1}_G) \leq m(|\pi|^2 f_{u,u}) = |\pi|^2(u,u)_{\mathcal{H}}$$

and taking the square roots yields $\|u\| \leq |\pi|\|u\|_{(\cdot,\cdot)_{\mathcal{H}}}$ for all $u \in \mathcal{H}$. We have then established that

$$\frac{1}{|\pi|}\|u\| \leq \|u\|_{(\cdot,\cdot)_{\mathcal{H}}} \leq |\pi|\|u\|$$

for any $u \in \mathcal{H}$, *i.e.* $\|\cdot\|$ and $\|\cdot\|_{(\cdot,\cdot)_{\mathcal{H}}}$ are equivalent norms on $\mathcal{H}$. Thus $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)_{\mathcal{H}}$ induce the same topology, and by Lemma 3.8 this implies that π is unitarisable. We conclude that G is unitarisable, and the proof is complete. □

Using results from Appendix B, we get new examples of unitarisable groups.

Corollary 3.14. Any solvable group is unitarisable.
In particular, $\mathbb{Z}^d$, $\mathbb{Q}^d$ and $\mathbb{R}^d$ are unitarisable, for any $d \geq 1$.

Proof. Combine Corollary B.20 and Theorem 3.13. □

Let us present now another example of application of Theorem 3.13.

Example 3.15. Let G be the *Heisenberg group* of 3×3 matrices with integer coefficients, *i.e.*

$$G := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{Z} \right\}.$$

The subgroup

$$N := \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, z \in \mathbb{Z} \right\}$$

is the center of G, and therefore is normal in G. Clearly $N \cong \mathbb{Z}$, and the quotient G/N is isomorphic to $\mathbb{Z}^2$ via the surjective group homomorphism

$$\begin{aligned} \varphi \colon G &\longrightarrow \mathbb{Z}^2 \\ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} &\longmapsto (x, y). \end{aligned}$$

Hence G is an extension of its center by $G/N \cong \mathbb{Z}^2$. As $\mathbb{Z}$ and $\mathbb{Z}^2$ are amenable, it follows that G is amenable, and in particular unitarisable by Theorem 3.13.

3.2 Non-unitarisability of non-abelian free groups

The first example of a non-unitarisable group, namely $\mathrm{SL}_2(\mathbb{R})$, was provided in 1955 by Ehrenpreis and Mautner [14]. We focus here on another example, the non-abelian free group on countably many generators. Coupled with stability properties proved earlier, this will imply that every group containing a non-abelian free subgroup, for instance $\mathrm{SL}_2(\mathbb{R})$, is not unitarisable.

The concept we introduce to construct uniformly bounded non-unitarisable representations of free groups is the one of a *derivation*.

Definition 3.16. Let π be a unitary representation of G.
A map $D\colon G \longrightarrow \mathcal{B}(\mathcal{H})$ is called a derivation with respect to π if

$$D(gh) = D(g)\pi(h) + \pi(g)D(h)$$

for all $g, h \in G$.

As for representations, a derivation D is called *bounded* if there exists $C > 0$ so that $\|D(g)\| \leq C$ for all $g \in G$.

Given a unitary representation π, it is possible to construct a wide set of derivations: pick $T \in \mathcal{B}(\mathcal{H})$, and define $D(g) := \pi(g)T - T\pi(g)$, $g \in G$. Indeed one has

$$\begin{aligned} D(gh) &= \pi(gh)T - T\pi(gh) \\ &= \pi(g)\pi(h)T - \pi(g)T\pi(h) + \pi(g)T\pi(h) - T\pi(g)\pi(h) \\ &= \pi(g)D(h) + D(g)\pi(h) \end{aligned}$$

for all $g, h \in G$. Such a derivation is called *inner*.

Given now π a unitary representation and a derivation D, one can produce another representation of G, by setting

$$\pi_D(g) := \begin{pmatrix} \pi(g) & D(g) \\ 0 & \pi(g) \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

Properties of π_D are now completely determined by those of D.

Proposition 3.17. Let π be a unitary representation of G, and $D\colon G \longrightarrow \mathcal{B}(\mathcal{H})$. The following equivalences hold.

(i) π_D is a representation of G if and only if D is a derivation.

(ii) π_D is uniformly bounded if and only if D is bounded.

(iii) π_D is unitarisable if and only if D is inner.

Proof. (i) We directly compute that

$$\pi_D(g)\pi_D(h) = \begin{pmatrix} \pi(g) & D(g) \\ 0 & \pi(g) \end{pmatrix} \begin{pmatrix} \pi(h) & D(h) \\ 0 & \pi(h) \end{pmatrix} = \begin{pmatrix} \pi(gh) & \pi(g)D(h) + D(g)\pi(h) \\ 0 & \pi(gh) \end{pmatrix}$$

using that π is a group homomorphism for the last equality. By definition, π_D is a representation of G if $\pi_D(gh) = \pi_D(g)\pi_D(h)$ for all $g, h \in G$, and this happens if and only if $D(gh) = \pi(g)D(h) + D(g)\pi(h)$ for all $g, h \in G$, *i.e.* if and only if D is a derivation.

(ii) Before proving the equivalence, we observe that

$$\|\pi_D(g)(u, v)\|_{\mathcal{H}\oplus\mathcal{H}} = \|(\pi(g)u + D(g)v, \pi(g)v)\|_{\mathcal{H}\oplus\mathcal{H}}$$

$$\begin{aligned}&\leq \|\pi(g)u + D(g)v\| + \|\pi(g)v\| \\ &\leq \|\pi(g)\|(\|u\| + \|v\|) + \|D(g)v\| \\ &\leq \sqrt{2}\|(u,v)\|_{\mathcal{H}\oplus\mathcal{H}} + \|D(g)v\|\end{aligned}$$

for all $g \in G$ and $u, v \in \mathcal{H}$, as π is unitary. Now we show the claimed equivalence.

To begin, assume that D is bounded by a constant $C > 0$. We have from the previous inequality that

$$\|\pi_D(g)(u,v)\|_{\mathcal{H}\oplus\mathcal{H}} \leq \sqrt{2}\|(u,v)\|_{\mathcal{H}\oplus\mathcal{H}} + C\|v\| \leq (C+\sqrt{2})\|(u,v)\|_{\mathcal{H}\oplus\mathcal{H}}$$

for all $g \in G$ and $u, v \in \mathcal{H}$. This implies $\|\pi_D(g)\| \leq C + \sqrt{2}$ for any $g \in G$, whence π_D is uniformly bounded.

Conversely, suppose that π_D is uniformly bounded by a constant $C > 0$. Then

$$\|D(g)v\| \leq \|\pi_D(g)(0,v)\|_{\mathcal{H}\oplus\mathcal{H}} \leq C\|(0,v)\|_{\mathcal{H}\oplus\mathcal{H}} = C\|v\|$$

for every $g \in G$ and $v \in \mathcal{H}$. Hence $\|D(g)\| \leq C$ for all $g \in G$, and D is bounded.

(iii) If D is inner, we may find $T \in \mathcal{B}(\mathcal{H})$ so that $D(g) = \pi(g)T - T\pi(g)$, for all $g \in G$. Consider then the operator

$$S := \begin{pmatrix} \mathrm{Id}_{\mathcal{H}} & T \\ 0 & \mathrm{Id}_{\mathcal{H}} \end{pmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$. It is a bounded operator, as

$$\begin{aligned}\|S(u,v)\|^2_{\mathcal{H}\oplus\mathcal{H}} &= \|(u+Tv, v)\|^2_{\mathcal{H}\oplus\mathcal{H}} \\ &= \|u+Tv\|^2 + \|v\|^2 \\ &\leq (\|u+Tv\| + \|v\|)^2 \\ &\leq (\|u\| + (\|T\|+1)\|v\|)^2 \\ &\leq 2(\|u\|^2 + (\|T\|+1)^2\|v\|^2) \\ &\leq 2(\|T\|+1)^2\|(u,v)\|^2_{\mathcal{H}\oplus\mathcal{H}}\end{aligned}$$

for all $u, v \in \mathcal{H}$, which yields to $\|S\| \leq 2(\|T\|+1)^2$. Moreover, we have

$$S^{-1}\pi_D(g)S = \begin{pmatrix} \mathrm{Id}_{\mathcal{H}} & -T \\ 0 & \mathrm{Id}_{\mathcal{H}} \end{pmatrix}\begin{pmatrix} \pi(g) & D(g) \\ 0 & \pi(g) \end{pmatrix}\begin{pmatrix} \mathrm{Id}_{\mathcal{H}} & T \\ 0 & \mathrm{Id}_{\mathcal{H}} \end{pmatrix} = \begin{pmatrix} \pi(g) & 0 \\ 0 & \pi(g) \end{pmatrix} = (\pi \oplus \pi)(g)$$

for any $g \in G$, and as π is unitary, $\pi \oplus \pi$ is unitary[15] as well, so π_D is unitarisable.

---

[15] Indeed, fix $g \in G$ and consider the operator $B(g)$ on $\mathcal{H}\oplus\mathcal{H}$ given by $B(g)(u,v) := (\pi(g)^*u, \pi(g)^*v)$, $(u,v) \in \mathcal{H} \oplus \mathcal{H}$. Then one has

$$\begin{aligned}\langle (u,v), B(g)(z,t)\rangle_{\mathcal{H}\oplus\mathcal{H}} &= \langle (u,v), (\pi(g)^*z, \pi(g)^*t)\rangle_{\mathcal{H}\oplus\mathcal{H}} \\ &= \langle u, \pi(g)^*z\rangle + \langle v, \pi(g)^*t\rangle \\ &= \langle \pi(g)u, z\rangle + \langle \pi(g)v, t\rangle \\ &= \langle (\pi\oplus\pi)(g)(u,v), (z,t)\rangle_{\mathcal{H}\oplus\mathcal{H}}\end{aligned}$$

for all $(u,v), (z,t) \in \mathcal{H}\oplus\mathcal{H}$, whence $(\pi\oplus\pi)(g)^* = B(g)$. As π is unitary, $B(g)$ equals in fact $(\pi\oplus\pi)(g^{-1})$, and thus $(\pi\oplus\pi)(g)^* = (\pi\oplus\pi)(g)^{-1}$ for all $g \in G$.

For the converse, assume π_D is unitarisable. Thus there is $S \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ so that $\tau(g) := S^{-1}\pi_D(g)S$ is a unitary operator on $\mathcal{H} \oplus \mathcal{H}$. In particular $\tau(g) = \tau(g^{-1})^*$ for any $g \in G$, and it follows that

$$\begin{aligned}(SS^*)\pi_D(g^{-1})^* &= (\pi_D(g^{-1})SS^*)^* \\ &= (S\tau(g^{-1})S^*)^* \\ &= S\tau(g^{-1})^*S^* \\ &= S\tau(g)S^* \\ &= \pi_D(g)(SS^*)\end{aligned}$$

for all $g \in G$. Writing explicitly

$$SS^* = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

we deduce from the above that $\pi(g)A_{12}+D(g)A_{22} = A_{12}\pi(g)$ and $\pi(g)A_{22} = A_{22}\pi(g)$, for all $g \in G$. Furthermore, note that as SS^* is positive and invertible, Corollary 1.23 provides $\varepsilon > 0$ so that

$$\langle SS^*(u, v), (u, v)\rangle_{\mathcal{H}\oplus\mathcal{H}} \geq \varepsilon\|(u, v)\|^2_{\mathcal{H}\oplus\mathcal{H}}$$

for all $(u, v) \in \mathcal{H} \oplus \mathcal{H}$. Expanding the inner product on the left-hand side, and restricting to vectors of the form $(0, v)$, $v \in \mathcal{H}$, one gets

$$\langle A_{22}v, v\rangle \geq \varepsilon\|v\|^2$$

for any $v \in \mathcal{H}$. In particular, A_{22} is positive and invertible as well. We deduce that

$$D(g) = (A_{12}\pi(g) - \pi(g)A_{12})A_{22}^{-1} = (A_{12}A_{22}^{-1})\pi(g) - \pi(g)(A_{12}A_{22}^{-1})$$

for any $g \in G$, proving that D is inner, as $A_{12}A_{22}^{-1} \in \mathcal{B}(\mathcal{H})$. This finishes the proof. □

Using this generic construction, we can now show the following.

Theorem 3.18. The group F_∞ is not unitarisable.

Proof. Let $G := F_\infty$ and let S be a countable generating set of G. Consider the linear operator A on $\ell^2(G)$ defined by

$$A(\delta_e) = 0,\ A(\delta_{s_1\ldots s_k}) = \delta_{s_1\ldots s_{k-1}}$$

for every reduced word $s_1 \ldots s_{k-1}s_k \in G$ (and e denotes the neutral element of G here). As $\{\delta_g : g \in G\}$ is an orthonormal basis of $\ell^2(G)$, this completely determines A. Moreover, $\|A(\delta_h)\|_2 \leq \|\delta_h\|_2$ for any $h \in G$, whence A is bounded with $\|A\| \leq 1$. Now, denote λ the regular representation of G, and define $D\colon G \longrightarrow \mathcal{B}(\ell^2(G))$ by

$$D(g) := A\lambda(g) - \lambda(g)A$$

for all $g \in G$. We first prove that D is a bounded derivation with respect to λ.

For all $g, h \in G$, one has

$$\begin{aligned} D(g)\lambda(h) + \lambda(g)D(h) &= (A\lambda(g) - \lambda(g)A)\lambda(h) + \lambda(g)(A\lambda(h) - \lambda(h)A) \\ &= A\lambda(gh) - \lambda(gh)A \\ &= D(gh) \end{aligned}$$

so D is a derivation with respect to λ. For the boundedness, note that

$$\|D(g)(\delta_h)\|_2 \leq \|A\| \|\lambda(g)(\delta_h)\|_2 + \|\lambda(g)\| \|A(\delta_h)\|_2 \leq 2$$

for any $h \in G$. We deduce then $\|D(g)\| \leq 2$ for any $g \in G$. Points (i) and (ii) of Proposition 3.17, which we may apply since λ is a unitary representation of G (by Example 3.2(iii)), ensure that λ_D is a uniformly bounded representation of G.

Towards a contradiction, suppose that D is inner, and let $T \in \mathcal{B}(\ell^2(G))$ be so that

$$D(g) = \lambda(g)T - T\lambda(g), \; g \in G.$$

For $g \in G$ write

$$T(\delta_g) = \sum_{h \in G} t(h, g)\delta_h$$

and set $p \colon G \longrightarrow \mathbb{C}$, $p(g) := \langle D(g)\delta_e, \delta_e \rangle$. Let $g, h \in G$ and observe that for any $x \in G$, we have

$$\lambda(g)(\delta_h)(x) = \delta_h(g^{-1}x) = \delta_{gh}(x)$$

whence the relation $\lambda(g)(\delta_h) = \delta_{gh}$ for all $g, h \in G$. We can thus compute

$$\begin{aligned} p(g) &= \langle \lambda(g)T(\delta_e) - T(\lambda(g)(\delta_e)), \delta_e \rangle \\ &= \left\langle \sum_{h \in G} t(h, e)\delta_{gh}, \delta_e \right\rangle - \langle T(\delta_g), \delta_e \rangle \\ &= t(g^{-1}, e) - t(e, g). \end{aligned}$$

The two functions $g \longmapsto t(g^{-1}, e)$, $g \longmapsto t(e, g)$ are in $\ell^2(G)$, as

$$\sum_{g \in G} t(e, g)^2 = \|T(\delta_e)\|^2 \leq \|T\|^2 < \infty$$

and

$$\sum_{g \in G} t(g^{-1}, e)^2 = \|T^*(\delta_e)\|^2 \leq \|T^*\|^2 = \|T\|^2 < \infty.$$

Thus $p \in \ell^2(G)$ as well. On the other hand, if $g = s_1 \dots s_k \neq e$ one has

$$p(g) = \langle A(\delta_g), \delta_e \rangle = \langle \delta_{s_1 \dots s_{k-1}}, \delta_e \rangle = \begin{cases} 1 & \text{if } s_1 \dots s_{k-1} = e \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } g \in S \\ 0 & \text{otherwise} \end{cases} = \mathbf{1}_S(g)$$

where the first equality follows from the definition of p and the fact that $A(\delta_e) = 0$. This shows the equality $p = \mathbf{1}_S$, and thus $\mathbf{1}_S \in \ell^2(G)$, which implies $|S| < \infty$. This is the desired contradiction, and hence D is not inner. Hence λ_D is not unitarisable, and thus $G = F_\infty$ is not unitarisable either. □

Therefore, a way to prove the non-unitarisability of a group is to show that this group contains F_∞ as a subgroup.

To detect non-abelian free subgroups in a group, the following lemma is crucial. It is often referred to as the *Ping-Pong lemma*.

Lemma 3.19. Let G be a group, generated by a subset S with $|S| \geq 2$. Suppose that G acts on a set X and that there exists a collection $(A_s)_{s\in S}$ of non-empty disjoint subsets of X so that

$$s^k \cdot A_t \subset A_s$$

for all $s \neq t \in S$ and $k \in \mathbb{Z} \setminus \{0\}$. Then G is a free group on S.

Proof. Let $g \in G$ and write it as a non-trivial reduced word

$$g = s_1^{k_1} \dots s_n^{k_n}$$

with $n \geq 1$, $s_1, \dots, s_n \in S$ and $k_1, \dots, k_n \in \mathbb{Z} \setminus \{0\}$. This is possible because S generates G. It is enough to prove that $g \neq e_G$ (see [4, corollary 1.8]). We distinguish three cases:

(i) $|S| \geq 3$.

(ii) $|S| = 2$ and $s_1 = s_n$.

(iii) $|S| = 2$ and $s_1 \neq s_n$.

If we are either in case (i) or (ii), we can pick $t \in S$ so that $t \neq s_1$ and $t \neq s_n$. It thus follows from the assumption applied n times that

$$\begin{aligned} g \cdot A_t &= (s_1^{k_1} \dots s_n^{k_n}) \cdot A_t \\ &= (s_1^{k_1} \dots s_{n-1}^{k_{n-1}}) \cdot (s_n^{k_n} \cdot A_t) \\ &\subset (s_1^{k_1} \dots s_{n-1}^{k_{n-1}}) \cdot A_{s_n} \\ &\subset \dots \\ &\subset s_1^{k_1} \cdot A_{s_2} \\ &\subset A_{s_1}. \end{aligned}$$

By hypothesis, we have $A_{s_1} \cap A_t = \emptyset$ whence $g \neq e_G$. If we are rather in case (iii) we conjugate g by $s_1^{k_1}$ and

$$s_1^{k_1} g s_1^{-k_1} = s_1^{2k_1} s_2^{k_2} \dots s_n^{k_n} s_1^{-k_1}$$

expresses $s_1^{k_1} g s_1^{-k_1}$ as a non-trivial reduced word. This word satisfies assumptions of case (ii) we just handled, so we deduce $s_1^{k_1} g s_1^{-k_1} \neq e_G$, and it follows that $g \neq e_G$. This completes the proof. □

A well-known application of this result is the following.

Example 3.20. Consider the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^2$, and the two matrices $S_1 := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $S_2 := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Moreover, let

$$A_{S_1} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ |x| > |y| \right\}, \ A_{S_2} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ |x| < |y| \right\}.$$

If $\begin{pmatrix} x \\ y \end{pmatrix} \in A_{S_2}$, then

$$|x + 2y| \geq ||2y| - |x|| \geq |2y| - |x| > 2|y| - |y| = |y|$$

by the second triangle inequality, whence $S_1 \begin{pmatrix} x \\ y \end{pmatrix} \in A_{S_1}$. Likewise, $S_2 A_{S_1} \subset A_{S_2}$. Clearly A_{S_1} and A_{S_2} are non-empty and disjoint, so we may apply Lemma 3.19 to obtain that $G = \langle S_1, S_2 \rangle$ is a free group in $\mathrm{SL}_2(\mathbb{Z})$, namely a copy of the non-abelian free group on two generators F_2.

The Ping-Pong lemma allows also to prove the following.

Corollary 3.21. Let F be a free group of rank 2, and $2 \leq n \leq \infty$. Then F contains a free subgroup of rank n.

Proof. Denote $\{a, b\}$ a free basis for F. Given $2 \leq n \leq \infty$, let $I = \{0, \ldots, n-1\}$ (resp. $I = \mathbb{N}$) and let G be the subgroup of F generated by $S := \{s_i : i \in I\}$ where

$$s_i := a^i b a^{-i}, \ i \in I.$$

Consider the natural action of G on F by left multiplication, and for each $i \in I$ denote $A_{s_i} \subset F$ the set of elements of F whose reduced form starts as $a^i b^h$, for some $h \in \mathbb{Z} \backslash \{0\}$. This definition imposes $A_{s_i} \cap A_{s_j} = \emptyset$ for $i \neq j \in I$, and moreover, if $k \in \mathbb{Z} \setminus \{0\}$, every element in $s_i^k \cdot A_{s_j}$ has a reduced form starting as $a^i b^k a^{-i} a^j b^h = a^i b^k a^{j-i} b^h$ for some $h \in \mathbb{Z} \setminus \{0\}$. This means that $s_i^k \cdot A_{s_j} \subset A_{s_i}$, for all distinct pairs $i, j \in I$. All hypotheses of Lemma 3.19 are fulfilled, and thus G is a free group of rank $|I|$ in F. □

These results combined with the non-unitarisability of F_∞ give us new examples of non-unitarisable groups.

Corollary 3.22. Any group containing F_2 as a subgroup is not unitarisable. In particular, F_d is not unitarisable for any $2 \leq d \leq \infty$, as well as $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{Z})$, $\mathrm{GL}_2(\mathbb{R})$.

Proof. As F_2 contains F_∞ as a subgroup, and the latter is non-unitarisable by Theorem 3.18, F_2 cannot be unitarisable either by (the contrapositive of) Proposition 3.12. This implies the non-unitarisability of any group containing F_2, in particular F_d for any $d \geq 2$. We derive the non-unitarisability of $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{Z})$, $\mathrm{GL}_2(\mathbb{R})$ from Example 3.20. □

We end this introduction to unitarisability by an overview of other known results, established recently, and open problems on the theme.

To begin, let us mention another stability property: virtual[(16)] unitarisability implies unitarisability [29, corollary 8.21]. Moreover, extensions (in the sense of Definition B.12) of unitarisable groups by *amenable* groups give rise to unitarisable groups [30, theorem 4.19]. On the other hand, it is not known whether extensions of unitarisable groups by unitarisable groups are still unitarisable. Also, it is unknown whether the directed union of unitarisable groups is unitarisable, but in contrast there is in fact a version of Corollary B.18 for unitarisability [26, corollary 0.11]:

Theorem. A group G is unitarisable if and only if all its countable subgroups are unitarisable.

In the opposite direction, until recently the only known examples of non-unitarisable groups contained non-abelian free groups. In 2008, Nicolas Monod and Inessa Epstein proved the existence of non-unitarisable torsion groups [16, corollary 1.6], and established several connections with the *first L^2–Betti number* of a group and the *cost* of a group. In 2009, Nicolas Monod and Narutaka Ozawa obtained in [23, theorem 1] the following major result:

Theorem. Let G be a group. The following are equivalent.

(i) G is amenable.

(ii) $A \wr G$[(17)] is unitarisable for all abelian groups A.

(iii) $A \wr G$ is unitarisable for some infinite abelian group A.

They derived from this theorem the non-unitarisability of many *Burnside groups* [23, theorem 2].

(16) A group G is called *virtually* unitarisable if it contains a unitarisable subgroup H with $[G : H] < \infty$. More generally, if P is a group-theoretic property, we say that G is *virtually–P* if it contains a finite index subgroup that has property P.

(17) The notation $A \wr G$ stands for the *wreath product* of A and G, defined as the semi-direct product

$$(\bigoplus_G A) \rtimes G$$

where G acts on the direct sum via $g \cdot (a_h)_{h\in G} := (a_{g^{-1}h})_{h\in G}$, $g \in G$, $(a_h)_{h\in G} \in \bigoplus_G A$.

3.3 Fixed points and smallest unitarisers

In this part, we establish a connection between unitarisable representations of a group G and fixed points for a natural action of the group on the space of positive invertible operators on a Hilbert space. Using the weak operator topology on the latter, we prove that any unitarisable representation of a group can be unitarised with an operator having minimal size.

The starting point is the following observation: if G is a group and $\pi\colon G \longrightarrow \mathrm{Aut}(\mathcal{H})$ is a representation of G on a Hilbert space $\mathcal{H}$, we can use the natural action of $\mathrm{Aut}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$ to define an action of G on $\mathcal{P}(\mathcal{H})$, letting

$$\theta_\pi\colon G \times \mathcal{P}(\mathcal{H}) \longrightarrow \mathcal{P}(\mathcal{H}),\ \theta_\pi(g,P) := \pi(g)P\pi(g)^*.$$

As proved in Chapter 2, this is a well-defined map, and as π is a group homomorphism, this is a group action. Indeed, $\theta_\pi(e_G,P) = \pi(e_G)P\pi(e_G)^* = \mathrm{Id}_{\mathcal{H}}P\mathrm{Id}_{\mathcal{H}}^* = P$ for any $P \in \mathcal{P}(\mathcal{H})$, and

$$\begin{aligned}\theta_\pi(gh,P) &= \pi(gh)P\pi(gh)^* \\ &= \pi(g)\pi(h)P\pi(h)^*\pi(g)^* \\ &= \pi(g)\theta_\pi(h,P)\pi(g)^* \\ &= \theta_\pi(g,\theta_\pi(h,P))\end{aligned}$$

for any $g,h \in G$ and $P \in \mathcal{P}(\mathcal{H})$. In particular, since the action of $\mathrm{Aut}(\mathcal{H})$ is isometric (Proposition 2.6) and respects the geodesics we defined (Lemma 2.8), the same is true for θ_π.

In the sequel, if $g \in G$ and $P \in \mathcal{P}(\mathcal{H})$ we also write $g \cdot P$ for $\theta_\pi(g,P) \in \mathcal{P}(\mathcal{H})$.

Our first lemma shows that unitarisers for π give rise to fixed points for this action, and vice-versa.

Lemma 3.23. If $S \in U(\pi)$, then SS^* is a fixed point of θ_π. Conversely, if T is a fixed point of θ_π, then $T^{1/2} \in U(\pi)$.

Proof. First, suppose that $S \in U(\pi)$. We already observed that SS^* is positive in Example 1.8, and its invertibility follows from that of S. Thus $SS^* \in \mathcal{P}(\mathcal{H})$. Now, as $S \in U(\pi)$, $S^{-1}\pi(g)S$ is unitary for all $g \in G$, and in particular

$$S^{-1}\pi(g)S(S^{-1}\pi(g)S)^* = \mathrm{Id}_{\mathcal{H}}$$

for all $g \in G$. This implies that $S^{-1}\pi(g)SS^*\pi(g)^*(S^{-1})^* = \mathrm{Id}_{\mathcal{H}}$, and multiplying from the left by S and from the right by S^*, we get

$$\pi(g)SS^*\pi(g)^* = SS^*$$

for all $g \in G$, so SS^* is a fixed point of θ_π.

Conversely, let T be a fixed point of θ_π, *i.e.*

$$\pi(g)T\pi(g)^* = T$$

for all $g \in G$. Writing $T = T^{1/2}T^{1/2}$, one has $T^{-1/2}\pi(g)T^{1/2}T^{1/2}\pi(g)^*T^{-1/2} = \mathrm{Id}_{\mathcal{H}}$ for any $g \in G$, whence

$$(T^{1/2})^{-1}\pi(g)T^{1/2}\big((T^{1/2})^{-1}\pi(g)T^{1/2}\big)^* = \mathrm{Id}_{\mathcal{H}}$$

for any $g \in G$. As $(T^{1/2})^{-1}\pi(g)T^{1/2}$ is invertible, we deduce that it is in fact unitary for all $g \in G$, and thus $T^{1/2} \in U(\pi)$. This concludes the proof. □

A direct consequence of this lemma is the following, about the properties of unitarisers.

Corollary 3.24. If π is unitarisable, then π has a positive invertible unitariser.

Proof. Since π is unitarisable, we can pick $S \in U(\pi)$. Then SS^* is a fixed point of θ_π, and thus $\sqrt{SS^*} \in U(\pi)$. As $\sqrt{SS^*}$ is positive and invertible by Theorem 1.42, the claim follows. □

The goal of the next result is to show we can always choose a unitariser with minimal size.

Proposition 3.25. Let π be a unitarisable representation of a group G. Then there exists $S_\pi \in U(\pi)$ so that

$$s(S_\pi) = \inf_{S \in U(\pi)} s(S).$$

Moreover, this unitariser can be chosen in $\mathcal{P}(\mathcal{H})$.

Proof. By Proposition 3.5(iv) and (the proof of) Corollary 3.24, given $S \in U(\pi)$, the operations

$$S \longrightarrow SS^* \longrightarrow \sqrt{SS^*}$$

produce a positive unitariser of π with same size as S. Hence, without loss of generality, we can restrict our search to positive unitarisers.

Next, by Lemma 3.23 and Proposition 3.5(iv), we have a size-squaring bijection

$$\begin{aligned} U(\pi) \cap \mathcal{P}(\mathcal{H}) &\longrightarrow \mathcal{P}(\mathcal{H})^G \\ S &\longmapsto S^2 \end{aligned}$$

so it is enough to find $T_\pi \in \mathcal{P}(\mathcal{H})^G$ with minimal size. Since $\mathcal{P}(\mathcal{H})^G$ is closed[(18)] under multiplication by strictly positive scalars, and that $s(\lambda S) = s(S)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ by Proposition 3.5(ii), it suffices to find T_π in

$$\mathcal{P}(\mathcal{H})^G_1 := \mathcal{P}(\mathcal{H})^G \cap \{A \in \mathcal{B}(\mathcal{H}) : \|A\| = 1\}.$$

We are thus left to prove there is $T_\pi \in \mathcal{P}(\mathcal{H})^G_1$ so that

$$s(T_\pi) = \inf_{T \in \mathcal{P}(\mathcal{H})^G_1} s(T).$$

Additionally, observe that if $\|T\| = 1$, $s(T) = \|T^{-1}\| = \frac{1}{\min_{\lambda \in \sigma(T)} \lambda}$, so in order to exhibit T_π realizing the infimum above, we must maximise the quantities $\min_{\lambda \in \sigma(T)} \lambda$ when T runs over $\mathcal{P}(\mathcal{H})^G_1$, *i.e.* exhibit a $T_\pi \in \mathcal{P}(\mathcal{H})^G_1$ so that

$$\min_{\lambda \in \sigma(T_\pi)} \lambda = \sup_{T \in \mathcal{P}(\mathcal{H})^G_1} \big(\min_{\lambda \in \sigma(T)} \lambda \big). \tag{8}$$

One last reduction can be made observing that $\min_{\lambda \in \sigma(T)} \lambda = 1 - \|\mathrm{Id}_\mathcal{H} - T\|$ for $\|T\| = 1$. Indeed, if $\|T\| = 1$, one has

$$\begin{aligned} 1 - \|\mathrm{Id}_\mathcal{H} - T\| &= 1 - \max_{\lambda \in \sigma(\mathrm{Id}_\mathcal{H} - T)} \lambda \\ &= 1 - \max_{\lambda \in \sigma(T)} (1 - \lambda) \\ &= 1 - (1 - \min_{\lambda \in \sigma(T)} \lambda) \\ &= \min_{\lambda \in \sigma(T)} \lambda \end{aligned}$$

using Lemma 1.16 in the second equality. With this observation, finding $T_\pi \in \mathcal{P}(\mathcal{H})^G_1$ verifying (8) is the same as finding $T_\pi \in \mathcal{P}(\mathcal{H})^G_1$ so that $\|\mathrm{Id}_\mathcal{H} - T_\pi\|$ is minimal and realizes the distance between $\mathrm{Id}_\mathcal{H}$ and $\mathcal{P}(\mathcal{H})^G_1$. Call then

$$\delta := \inf_{T \in \mathcal{P}(\mathcal{H})^G_1} \|\mathrm{Id}_\mathcal{H} - T\|$$

this distance, and note that $\delta < 1$ as

$$1 - \|\mathrm{Id}_\mathcal{H} - T\| = \min_{\lambda \in \sigma(T)} \lambda > 0$$

for any $T \in \mathcal{P}(\mathcal{H})^G_1$.

[(18)]Indeed, if $T \in \mathcal{P}(\mathcal{H})^G$ and $\lambda > 0$, then $\lambda T \in \mathcal{P}(\mathcal{H})$ by Lemma 2.1, and

$$\theta_\pi(g, \lambda T) = \pi(g)(\lambda T)\pi(g)^* = \lambda(\pi(g)T\pi(g)^*) = \lambda T$$

for all $g \in G$. Thus $\lambda T \in \mathcal{P}(\mathcal{H})^G$.

Pick a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathcal{H})_1^G$ so that $\|\mathrm{Id}_{\mathcal{H}} - T_n\| \longrightarrow \delta$ as $n \to \infty$. Now, $\mathcal{P}(\mathcal{H})_1^G \subset \{A \in \mathcal{B}(\mathcal{H}) : \|A\| \leq 1\}$ and this set is compact for the weak operator topology by Theorem 1.52. As this topology is metrisable on bounded parts of $\mathcal{B}(\mathcal{H})$ (Theorem 1.51), $\{A \in \mathcal{B}(\mathcal{H}) : \|A\| \leq 1\}$ is in fact sequentially compact by Theorem A.37. We thus extract from $(T_n)_{n\in\mathbb{N}}$ a convergent subsequence $(T_{\varphi(n)})_{n\in\mathbb{N}}$ and we denote T_π its limit. We now check that T_π is in $\mathcal{P}(\mathcal{H})_1^G$ and that $\delta = \|\mathrm{Id}_{\mathcal{H}} - T_\pi\|$.

First of all, as $\{A \in \mathcal{B}(\mathcal{H}) : \|A\| \leq 1\}$ is compact for τ_w, which is a Hausdorff topology (Lemma 1.50), Proposition A.32 ensures that $\{A \in \mathcal{B}(\mathcal{H}) : \|A\| \leq 1\}$ is also closed for τ_w, so we must have $T_\pi \in \{A \in \mathcal{B}(\mathcal{H}) : \|A\| \leq 1\}$, and thus $\|T_\pi\| \leq 1$.

Next, by definition of the weak operator topology, we have

$$\langle T_\pi u, u\rangle = \lim_{n\to\infty} \langle T_{\varphi(n)} u, u\rangle$$

for all $u \in \mathcal{H}$. As $T_{\varphi(n)} \in \mathcal{P}(\mathcal{H})$ for all $n \in \mathbb{N}$, $(\langle T_{\varphi(n)} u, u\rangle)_{n\in\mathbb{N}}$ is a sequence of positive real numbers, so its limit is a positive number. Hence T_π is positive.

Additionally, since $T_\pi - \mathrm{Id}_{\mathcal{H}}$ is self-adjoint, we have

$$\begin{aligned}
\|T_\pi - \mathrm{Id}_{\mathcal{H}}\| &= \sup_{\|u\|=1} |\langle (T_\pi - \mathrm{Id}_{\mathcal{H}})u, u\rangle| \\
&= \sup_{\|u\|=1} |\lim_{n\to\infty} \langle (T_{\varphi(n)} - \mathrm{Id}_{\mathcal{H}})u, u\rangle| \\
&= \sup_{\|u\|=1} \lim_{n\to\infty} |\langle (T_{\varphi(n)} - \mathrm{Id}_{\mathcal{H}})u, u\rangle| \\
&\leq \sup_{\|u\|=1} \lim_{n\to\infty} \|T_{\varphi(n)} - \mathrm{Id}_{\mathcal{H}}\| \\
&= \sup_{\|u\|=1} \delta \\
&= \delta
\end{aligned}$$

where the first equality follows from [13, theorem 1.12] (or [6, theorem 2.2.13]), and the upper bound follows from Cauchy-Schwarz inequality. In particular, this implies that $\sigma(T_\pi - \mathrm{Id}_{\mathcal{H}}) \subset [-\delta, \delta]$, and Lemma 1.16 in turn implies

$$\sigma(T_\pi) \subset [1 - \delta, \delta + 1] \subset (0, \delta + 1].$$

Thus T_π is invertible, so $T_\pi \in \mathcal{P}(\mathcal{H})$ and in fact, as we already know $\|T_\pi\| \leq 1$, we have $\sigma(T_\pi) \subset [1 - \delta, 1]$.

Now let $u \in \mathcal{H}$ and $g \in G$. We compute that

$$\begin{aligned}
\langle (g \cdot T_\pi)u, u\rangle &= \langle \pi(g) T_\pi \pi(g)^* u, u\rangle \\
&= \langle T_\pi \pi(g)^* u, \pi(g)^* u\rangle \\
&= \lim_{n\to\infty} \langle T_{\varphi(n)} \pi(g)^* u, \pi(g)^* u\rangle \\
&= \lim_{n\to\infty} \langle (g \cdot T_{\varphi(n)})u, u\rangle \\
&= \lim_{n\to\infty} \langle T_{\varphi(n)} u, u\rangle
\end{aligned}$$

$$= \langle T_\pi u, u\rangle$$

using the fact that $T_{\varphi(n)}$ is a fixed point for $G \curvearrowright \mathcal{P}(\mathcal{H})$ and the definition of the weak operator topology. This shows that

$$\langle (g \cdot T_\pi - T_\pi)u, u\rangle = 0$$

for all $u \in \mathcal{H}$, and Lemma 1.3 ensures then that $g \cdot T_\pi = T_\pi$. As this holds for any $g \in G$, we conclude that T_π is a fixed point for $G \curvearrowright \mathcal{P}(\mathcal{H})$.

To finish, assume towards a contradiction that $b := \|T_\pi\| < 1$. Letting $a := \min_{\lambda\in\sigma(T_\pi)} \sigma(T_\pi)$, we have then

$$\sigma(T_\pi) \subset [a, b],\ 0 < 1 - \delta \leq a < b < 1.$$

The operator $T := \frac{1}{b}T_\pi$ is therefore positive, invertible, fixed by θ_π, and of norm 1. In other words, it lies in $\mathcal{P}(\mathcal{H})_1^G$, and thus $\|\mathrm{Id}_\mathcal{H} - T\| \geq \delta$. On the other hand, $\sigma(T) \subset [\frac{a}{b}, 1]$, which implies

$$\|\mathrm{Id}_\mathcal{H} - T\| \leq 1 - \frac{a}{b} < 1 - a = \|\mathrm{Id}_\mathcal{H} - T_\pi\| \leq \delta$$

whence $\|\mathrm{Id}_\mathcal{H} - T\| < \delta$. This contradicts $\|\mathrm{Id}_\mathcal{H} - T\| \geq \delta$, so we must have $b \geq 1$. Hence $\|T_\pi\| = 1$. We deduce now that $T_\pi \in \mathcal{P}(\mathcal{H})_1^G$, and in particular

$$\|\mathrm{Id}_\mathcal{H} - T_\pi\| \geq \delta.$$

As we already proved that $\|\mathrm{Id}_\mathcal{H} - T_\pi\| \leq \delta$, we conclude that $\|\mathrm{Id}_\mathcal{H} - T_\pi\| = \delta$, so T_π indeed realizes the distance between $\mathcal{P}(\mathcal{H})_1^G$ and $\mathrm{Id}_\mathcal{H}$. As explained above, this concludes the proof. □

A key idea towards Pisier's result is that, for unitarisable groups, we can always have a good control on sizes of representations, and we will show in the proof that if a representation has large size, we can always "deform" it in a continuous way to get representations with smaller sizes. The next definition introduces this deformation.

Definition 3.26. Let π be a unitarisable representation of a group G, with $S \in U(\pi)$ a smallest positive unitariser. For $t \in [0, 1]$, let

$$\begin{aligned} \pi_t \colon G &\longrightarrow \mathrm{Aut}(\mathcal{H}) \\ g &\longmapsto S^{-t}\pi(g)S^t. \end{aligned}$$

For $t \in [0, 1]$, π_t is a representation of G, as

$$\begin{aligned} \pi_t(gh) &= S^{-t}\pi(gh)S^t \\ &= S^{-t}\pi(g)S^tS^{-t}\pi(h)S^t \\ &= \pi_t(g)\pi_t(h) \end{aligned}$$

for all $g, h \in G$.

Knowing that S is a smallest unitariser for π allows one to exhibit a smallest unitariser for π_t.

Lemma 3.27. Let π be a unitarisable representation of a group G, and let $S \in U(\pi)$ be a smallest positive unitariser.
Then for any $t \in [0,1]$, S^{1-t} is a smallest unitariser for π_t.

Proof. Fix $t \in [0,1]$. For $g \in G$, we have

$$(S^{1-t})^{-1}\pi_t(g)S^{1-t} = S^{t-1}S^{-t}\pi(g)S^tS^{1-t} = S^{-1}\pi(g)S \in \mathcal{U}(\mathcal{H})$$

as S unitarises π. Hence $S^{1-t} \in U(\pi_t)$. Up to normalizing S, and invoking Proposition 3.5(ii) and the fact that $U(\pi)$ is closed under scaling, we may assume that $\|S\| = 1$. Towards a contradiction, suppose there exists $Q \in U(\pi_t)$ with

$$s(Q) < s(S^{1-t}).$$

As for S, we may assume that $\|Q\| = 1$. Since $\|S\| = 1$, $\|S^{1-t}\| = 1$ using Theorem 1.30(i), so the assumption $s(Q) < s(S^{1-t})$ reads as

$$\|Q^{-1}\| < \|S^{t-1}\|.$$

Now, we claim that $S^tQ \in U(\pi)$. Indeed, it suffices to write

$$(S^tQ)^{-1}\pi(g)S^tQ = Q^{-1}S^{-t}\pi(g)S^tQ = Q^{-1}\pi_t(g)Q$$

for any $g \in G$, and by assumption Q unitarises π_t, so $Q^{-1}\pi_t(g)Q$ is unitary for any $g \in G$. Now, looking at the size of S^tQ provides

$$s(S^tQ) = \|S^tQ\|\|(S^tQ)^{-1}\| \leq \|S^t\|\|Q\|\|Q^{-1}\|\|S^{-t}\| < \|S^t\|\|S^{t-1}\|\|S^{-t}\|.$$

Additionally, using once again Theorem 1.30, one gets $\|S^t\| = 1$, as well as

$$\|S^{t-1}\| = \|(S^{-1})^{1-t}\| = \|S^{-1}\|^{1-t} = s(S)^{1-t}$$

and $\|S^{-t}\| = \|(S^{-1})^t\| = \|S^{-1}\|^t = s(S)^t$. Putting this in our previous estimate, it follows that

$$s(S^tQ) < \|S^t\|\|S^{t-1}\|\|S^{-t}\| = s(S)^{1-t}s(S)^t = s(S).$$

This contradicts the fact that S is a smallest unitariser for π. We conclude that such a Q cannot exist, and thus S^{1-t} is a smallest unitariser for π_t. We are done. □

Actually, the proof also provides a formula for the size of the smallest unitariser we just found.

Corollary 3.28. Let π be a unitarisable representation of G, and $S \in U(\pi)$ be a smallest positive unitariser for π. Then

$$s(S^{1-t}) = s(S)^{1-t}.$$

Proof. Using once again Theorem 1.30(i), we compute

$$s(S^{1-t}) = \|S^{1-t}\|\|(S^{-1})^{1-t}\| = \|S\|^{1-t}\|S^{-1}\|^{1-t} = s(S)^{1-t}$$

as announced. □

3.4 Metric properties of uniformly bounded representations

This part is devoted to the interplay between uniformly bounded representations of a group and the metric structure on $\mathcal{P}(\mathcal{H})$ we defined in Chapter 2.

We start by recording the following.

Lemma 3.29. Let π be a representation of a group G, and let θ_π be the induced action of G on $\mathcal{P}(\mathcal{H})$. Then the set $\mathcal{P}(\mathcal{H})^G$ is metrically convex.

Proof. Let $A, B \in \mathcal{P}(\mathcal{H})^G$. Since the action of G preserves the geodesic between A and B, we have

$$g \cdot \sigma(A, B, t) = \sigma(g \cdot A, g \cdot B, t) = \sigma(A, B, t)$$

for any $g \in G$ and $t \in [0, 1]$. Thus $\sigma(A, B, t) \in \mathcal{P}(\mathcal{H})^G$. □

Furthermore, if π is uniformly bounded we have $|\pi|^2 \geq \|\pi(g)\|^2 = \|\pi(g)\pi(g)^*\|$ and likewise $|\pi|^2 \geq \|(\pi(g)\pi(g)^*)^{-1}\|$, for any $g \in G$. Thus the orbit of $\mathrm{Id}_\mathcal{H} \in \mathcal{P}(\mathcal{H})$ under $\theta_\pi \colon G \times \mathcal{P}(\mathcal{H}) \longrightarrow \mathcal{P}(\mathcal{H})$ is bounded, as

$$\begin{aligned} \mathrm{d}(\theta_\pi(g, \mathrm{Id}_\mathcal{H}), \mathrm{Id}_\mathcal{H}) &= \| \ln(\pi(g)\pi(g)^*)\| \\ &= \max(\ln(\|\pi(g)\pi(g)^*\|), \ln(\|(\pi(g)\pi(g)^*)^{-1}\|)) \\ &\leq \ln(|\pi|^2) \end{aligned}$$

for all $g \in G$, using Lemma 2.5. It therefore makes sense to look at the diameter of the orbit of $\mathrm{Id}_\mathcal{H}$.

Definition 3.30. Let π be a uniformly bounded representation of a group G. Its diameter is the diameter of the orbit of $\mathrm{Id}_\mathcal{H}$ under the action θ_π, namely

$$\mathrm{diam}(\pi) := \sup_{g,h \in G} \mathrm{d}(g \cdot \mathrm{Id}_\mathcal{H}, h \cdot \mathrm{Id}_\mathcal{H}).$$

Since G acts by isometries on $\mathcal{P}(\mathcal{H})$, this can also be written as

$$\sup_{g\in G} \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, g \cdot \mathrm{Id}_{\mathcal{H}}).$$

This observation allows us to easily compute the diameter of a representation in term of its size.

Proposition 3.31. Let π be a uniformly bounded representation of a group G. Then one has

$$\mathrm{diam}(\pi) = 2\ln(|\pi|).$$

Proof. Using Definition 3.30 and Lemma 2.5, we have

$$\begin{aligned}
\mathrm{diam}(\pi) &= \sup_{g\in G} \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, g \cdot \mathrm{Id}_{\mathcal{H}}) \\
&= \sup_{g\in G} \| \ln(\pi(g)\pi(g)^*)\| \\
&= \sup_{g\in G} \max(\ln(\|\pi(g)\pi(g)^*\|), \ln(\|(\pi(g)\pi(g)^*)^{-1}\|)) \\
&= \sup_{g\in G} \max(\ln(\|\pi(g)\|^2), \ln(\|\pi(g)^{-1}\|^2)) \\
&= 2\sup_{g\in G} \max(\ln(\|\pi(g)\|), \ln(\|\pi(g^{-1})\|)) \\
&= 2\sup_{g\in G} \ln(\|\pi(g)\|) \\
&= 2\ln(\sup_{g\in G} \|\pi(g)\|) \\
&= 2\ln(|\pi|)
\end{aligned}$$

as we wanted. The proof is complete. □

Beyond its straightforward proof, this result really relates size of a representation of G with the way on which the orbit of $\mathrm{Id}_{\mathcal{H}}$ sits inside $\mathcal{P}(\mathcal{H})$. In words, better is the control on $|\pi|$, closer to $\mathrm{Id}_{\mathcal{H}}$ is its orbit.

With this explicit formula for the diameter, we can then establish an analog to Corollary 3.28 for the size of representations.

Proposition 3.32. Let π be a unitarisable representation of a group G, with a smallest positive unitariser $S \in U(\pi)$. Then

$$|\pi_t| \leq |\pi|^{1-t}$$

for any $t \in [0, 1]$.

Proof. Let $t \in [0,1]$. Equivalently, we establish $2\ln(|\pi_t|) \leq 2\ln(|\pi|^{1-t})$. Appealing Proposition 3.31, we compute

$$\begin{aligned}
2\ln(|\pi_t|) = \operatorname{diam}(\pi_t) &= \sup_{g\in G} \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, \pi_t(g)\pi_t(g)^*) \\
&= \sup_{g\in G} \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, S^{-t}\pi(g)S^{2t}\pi(g)^*S^{-t}) \\
&= \sup_{g\in G} \mathrm{d}(S^{2t}, \pi(g)S^{2t}\pi(g)^*) \\
&= \sup_{g\in G} \mathrm{d}(S^{2t}, g\cdot S^{2t}) \\
&= \sup_{g\in G} \mathrm{d}(\sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t), g\cdot\sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t)) \\
&= \sup_{g\in G} \mathrm{d}(\sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t), \sigma(g\cdot\mathrm{Id}_{\mathcal{H}}, g\cdot S^2, t)) \\
&\leq \sup_{g\in G}((1-t)\mathrm{d}(\mathrm{Id}_{\mathcal{H}}, g\cdot\mathrm{Id}_{\mathcal{H}}) + t\mathrm{d}(S^2, g\cdot S^2))
\end{aligned}$$

using definitions of the diameter and of π_t, the definition of the action of G on $\mathcal{P}(\mathcal{H})$, the fact that this action preserves d and the geodesics, and Theorem 2.23 for the last inequality. Now since $S \in U(\pi)$ and is self-adjoint, Lemma 3.23 ensures that $SS^* = S^2$ is a fixed point of θ_π, so $g \cdot S^2 = S^2$ for all $g \in G$, and the second term in the last supremum vanishes. We are left with

$$(1-t)\sup_{g\in G} \mathrm{d}(\mathrm{Id}_{\mathcal{H}}, g\cdot\mathrm{Id}_{\mathcal{H}}) = (1-t)\operatorname{diam}(\pi) = 2(1-t)\ln(|\pi|) = 2\ln(|\pi|^{1-t})$$

and thus $2\ln(|\pi_t|) \leq 2\ln(|\pi|^{1-t})$. This completes the proof. □

For the next result, we need a tool from real analysis. Recall that if $J \subset \mathbb{R}$ is an interval and $\mathcal{F} = \{f_i : J \longrightarrow \mathbb{R}\}_{i\in I}$ is a (not necessarily countable) family of functions defined on J, $\mathcal{F}$ is called *uniformly equicontinuous* if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall i \in I,\ \forall t, t' \in J,\ |t - t'| < \delta \Longrightarrow |f_i(t) - f_i(t')| < \varepsilon.$$

Lemma 3.33. If $\mathcal{F} = \{f_i : J \longrightarrow \mathbb{R}\}_{i\in I}$ is uniformly equicontinuous, the function

$$\begin{aligned}
g : J &\longrightarrow \mathbb{R} \\
t &\longmapsto \sup_{i\in I} f_i(t)
\end{aligned}$$

is continuous on J.

Proof. Let $\varepsilon > 0$. By assumption, there exists $\delta > 0$, depending only on ε, so that

$$\forall i \in I,\ \forall t, t' \in J,\ |t - t'| < \delta \Longrightarrow |f_i(t) - f_i(t')| < \varepsilon.$$

Fix then $t, t' \in J$ with $|t - t'| < \delta$. By the above, $f_i(t) < f_i(t') + \varepsilon$ for all $i \in I$, whence $g(t) < g(t') + \varepsilon$. Exchanging the role of t and t' provides $g(t') < g(t) + \varepsilon$, and thus $|g(t) - g(t')| < \varepsilon$. This shows that g is continuous, as wanted. □

In fact, note that our proof even provides the uniform continuity of g.

Additionally, note that if (X, d_X) is a metric space, then

$$|\mathrm{d}_X(x_1, y_2) - \mathrm{d}_X(x_2, y_1)| \leq \mathrm{d}_X(x_1, x_2) + \mathrm{d}_X(y_1, y_2) \tag{9}$$

for all $x_1, x_2, y_1, y_2 \in X$. Indeed, the triangle inequality implies first that

$$\mathrm{d}_X(x_1, y_2) \leq \mathrm{d}_X(x_1, x_2) + \mathrm{d}_X(x_2, y_1) + \mathrm{d}_X(y_1, y_2)$$

whence $\mathrm{d}_X(x_1, y_2) - \mathrm{d}_X(x_2, y_1) \leq \mathrm{d}_X(x_1, x_2) + \mathrm{d}_X(y_1, y_2)$. In the same way, one gets

$$\mathrm{d}_X(x_2, y_1) - \mathrm{d}_X(x_1, y_2) \leq \mathrm{d}_X(x_1, x_2) + \mathrm{d}_X(y_1, y_2)$$

yielding

$$|\mathrm{d}_X(x_1, y_2) - \mathrm{d}_X(x_2, y_1)| \leq \mathrm{d}_X(x_1, x_2) + \mathrm{d}_X(y_1, y_2).$$

We will use these two facts in the proof of the proposition below.

Proposition 3.34. Let π be a unitarisable representation of a group G with a smallest positive unitariser $S \in U(\pi)$. Then the function

$$\begin{aligned} [0,1] &\longrightarrow \mathbb{R} \\ t &\longmapsto |\pi_t| \end{aligned}$$

is continuous.

Proof. From the proof of Proposition 3.32, we have

$$2\ln(|\pi_t|) = \sup_{g \in G} \mathrm{d}(\sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t), g \cdot \sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t))$$

so it is enough to prove that the right hand side is a continuous function on $[0,1]$. Invoking Lemma 3.33, it is enough to show that the family of functions over which we take the supremum is uniformly equicontinuous. Let then $\varepsilon > 0$, and set $\delta := \frac{\varepsilon}{4\|\ln(S)\|} > 0$. Let $g \in G$, and fix $t, t' \in [0,1]$ with $|t - t'| < \delta$. By (9), it follows that

$$\begin{aligned}
&\left|\mathrm{d}(\sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t), g \cdot \sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t)) - \mathrm{d}(\sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t'), g \cdot \sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t'))\right| \\
&\leq \mathrm{d}(\sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t), \sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t')) + \mathrm{d}(g \cdot \sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t), g \cdot \sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t')) \\
&= 2\mathrm{d}(\sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t), \sigma(\mathrm{Id}_{\mathcal{H}}, S^2, t')) \\
&= 2\|\ln(S^{2(t'-t)})\| \\
&= 4\|\ln(S)\||t' - t|
\end{aligned}$$

$< \varepsilon$

using the invariance of d under the action of G for the first equality. The second one is the definition of the metric d, and the third one is Remark 1.40. We deduce, as explained above, that $t \longmapsto 2\ln(|\pi_t|)$ is continuous on $[0,1]$, so $t \longmapsto |\pi_t|$ is continuous on $[0,1]$ as well. □

3.5 Proof of Pisier's theorem

In this final part, we make use of all the framework developed above to establish Pisier's result, and translate it completely geometrically. At the end, this provides another characterization of amenability.

The first key observation is the next proposition.

Proposition 3.35. Let G be a unitarisable group. Let $(\pi_n)_{n\in\mathbb{N}}$ be a family of uniformly bounded representations, so that

$$\sup_{n\in\mathbb{N}} |\pi_n| < \infty.$$

Then there exists a constant $C > 0$ and a collection $(S_n)_{n\in\mathbb{N}}$ with $S_n \in U(\pi_n)$ so that $s(S_n) \leq C$ for any $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$, let $\pi_n : G \longrightarrow \mathrm{Aut}(\mathcal{H}_n)$ be a uniformly bounded representation of G on a Hilbert space $\mathcal{H}_n$. Now, consider the representation

$$\pi := \bigoplus_{n\in\mathbb{N}} \pi_n$$

on $\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$. As $\sup_{n\in\mathbb{N}} |\pi_n| < \infty$, π is a uniformly bounded representation of G. As the latter is unitarisable, there exists $S \in \mathrm{Aut}\big(\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n\big)$ so that $S^{-1}\pi(g)S$ is unitary for every $g \in G$. In particular, for each $n \in \mathbb{N}$, $S|_{\mathcal{H}_n}\pi_n(g)(S^{-1})|_{S(\mathcal{H}_n)}$ is unitary for every $g \in G$. Fixing an arbitrary unitary equivalence $U : S(\mathcal{H}_n) \longrightarrow \mathcal{H}_n$, it follows that $(US|_{\mathcal{H}_n})^{-1} : \mathcal{H}_n \longrightarrow \mathcal{H}_n$ unitarises π_n, for any $n \in \mathbb{N}$. We thus set

$$S_n := (US|_{\mathcal{H}_n})^{-1}$$

for all $n \in \mathbb{N}$, and it follows that

$$\begin{aligned} s(S_n) &= s((US|_{\mathcal{H}_n})^{-1}) \\ &= \|(US|_{\mathcal{H}_n})^{-1}\| \|US|_{\mathcal{H}_n}\| \end{aligned}$$

$$\begin{aligned}
&= \|S^{-1}|_{S(\mathcal{H}_n)}\| \|S|_{\mathcal{H}_n}\| \\
&\leq \|S\| \|S^{-1}\| \\
&= s(S)
\end{aligned}$$

using that U and U^{-1} are unitary. The statement follows setting $C := s(S) > 0$. □

We can now state and establish one of the main goals of this thesis.

Theorem 3.36. Let G be a unitarisable group.
There exist universal constants $\alpha, K > 0$, depending only on G, so that for every uniformly bounded representation π of G, there exists $S \in U(\pi)$ with

$$s(S) \leq K|\pi|^{\alpha}.$$

Proof. Fix a unitarisable group G. Towards a contradiction, we suppose that the negation of the claim holds, *i.e.* for every $\alpha, K > 0$, there exists a uniformly bounded representation $\pi_{\alpha,K}$ of G so that for every $S \in U(\pi)$, $s(S) > K|\pi|^{\alpha}$. In particular, letting $n \in \mathbb{N}$ and choosing $\alpha = K = n$ provides a Hilbert space $\mathcal{H}_n$ and a uniformly bounded representation $\pi_n : G \longrightarrow \mathrm{Aut}(\mathcal{H}_n)$ so that

$$s(S) > n|\pi_n|^n$$

for all $S \in U(\pi_n)$. By Proposition 3.25, for all $n \in \mathbb{N}$, we may pick a smallest unitariser $S_n \in U(\pi_n)$, and $s(S_n) > n|\pi_n|^n$.

We are now going to show that the sequence $(\pi_n)_{n\in\mathbb{N}}$ can be chosen with two properties:

$$\text{(i) } \forall n \in \mathbb{N},\ |\pi_n| \leq 2 \text{ and (ii) } \forall n \in \mathbb{N},\ s(S_n) > n.$$

First of all, assume that for a fixed $n \in \mathbb{N}$, $|\pi_n| > 2$. Consider then the representation $\pi_{n,t} : G \longrightarrow \mathrm{Aut}(\mathcal{H}_n)$ defined as

$$\pi_{n,t}(g) := S_n^{-t} \pi_n(g) S_n^t$$

for every $g \in G$. The function $t \longmapsto |\pi_{n,t}|$ is continuous on $[0,1]$ by Proposition 3.34, and takes value $|\pi_{n,0}| = |\pi_n| > 2$ at $t = 0$ and value $|\pi_{n,1}| = 1$ at $t = 1$ since $S_n \in U(\pi_n)$. By the intermediate value theorem, there exists $t_n \in (0,1)$ so that $|\pi_{n,t_n}| = 2$. For π_{n,t_n}, Lemma 3.27 provides a smallest unitariser $S_n^{1-t_n}$, whose size satisfies

$$s(S_n^{1-t_n}) = s(S_n)^{1-t_n} > (n|\pi_n|^n)^{1-t_n} \geq n^{1-t_n}|\pi_{n,t_n}|^n \geq 2^n > n$$

using Corollary 3.28 for the first equality. The first inequality comes from the assumption on $s(S_n)$, and the second is Proposition 3.32. Henceforth, in our initial sequence, we can replace π_n by π_{n,t_n} and this representation now satisfies both (i) and (ii).

On the other hand if $|\pi_n| \leq 2$ (so (i) already holds), it is enough to observe that $|\pi_n| \geq 1$ to get

$$s(S_n) > n|\pi_n|^n \geq n$$

and thus (ii) holds as well.

Hence we just proved that without restriction, we may assume that our sequence $(\pi_n)_{n\in\mathbb{N}}$ satisfies (i) and (ii) above. We are now in position to apply Proposition 3.35, and we get a constant $C > 0$ and a family $(T_n : \mathcal{H}_n \longrightarrow \mathcal{H}_n)_{n\in\mathbb{N}}$ of bounded operators, so that T_n unitarises π_n, with $s(T_n) \leq C$ for all $n \in \mathbb{N}$. Now it follows

$$C \geq s(T_n) \geq s(S_n) > n$$

for any $n \in \mathbb{N}$, which is absurd. Thus there must exist $\alpha, K > 0$ so that, for any uniformly bounded representation π, there is $S \in U(\pi)$ with

$$s(S) \leq K|\pi|^{\alpha}.$$

This concludes the proof. □

As we saw in the proof of Proposition 3.25, the existence of a smallest unitariser for a unitarisable representation π of the group G is equivalent to the existence of a fixed point of θ_π of norm 1 realizing the distance between $\mathrm{Id}_{\mathcal{H}}$ and $\mathcal{P}(\mathcal{H})_1^G$.

We now show that we can choose this fixed point conveniently in order to relate its distance to $\mathrm{Id}_{\mathcal{H}}$ with the size of a smallest unitariser.

To this aim, we start by noticing that if a self-adjoint operator $T \in \mathcal{B}(\mathcal{H})$ has *spectral symmetry*, in the sense that

$$\max_{\lambda\in\sigma(T)} \lambda = \frac{1}{\min_{\lambda\in\sigma(T)} \lambda}$$

then (i) $s(T) = \|T\|^2$ and, if in addition T is positive, (ii) $\|\ln(T)\| = \ln(\|T\|)$.

Indeed, having spectral symmetry simply means that $\|T\| = \|T^{-1}\|$, so $s(T) = \|T\|\|T^{-1}\| = \|T\|^2$. If additionally T is positive, we directly get

$$\|\ln(T)\| = \max(\ln(\|T\|), \ln(\|T^{-1}\|)) = \ln(\|T\|)$$

by Lemma 2.5.

We use this observation to prove two auxiliary lemmas.

Lemma 3.37. Let π be a unitarisable representation of a group G. If $S \in U(\pi)$, then there is $T' \in \mathcal{P}(\mathcal{H})^G$ so that

$$\mathrm{d}(T', \mathrm{Id}_{\mathcal{H}}) = \ln(s(S)).$$

Proof. Fix then S a unitariser of π. Multiplying S by $\lambda := \sqrt{\frac{\|S^{-1}\|}{\|S\|}}$, we get an invertible operator $S' = \lambda S$ that still unitarises π, that has the same size as S by Proposition

3.5(ii), and which furthermore has spectral symmetry. By Lemma 3.23, we get a fixed point
$$T' = S'(S')^* = (\lambda S)(\lambda S)^* = \lambda^2 SS^*$$
which has also spectral symmetry, because of the C^*−identity and the spectral symmetry of S':
$$\|T'\| = \|S'(S')^*\| = \|S'\|^2 = \|S'^{-1}\|^2 = \|(S')^{-1}(S'^{-1})^*\| = \|T'^{-1}\|.$$
Additionally, note that $\|T'\| = \lambda^2\|SS^*\| = \frac{\|S^{-1}\|}{\|S\|}\|S\|^2 = \|S\|\|S^{-1}\| = s(S)$ and thus it follows that
$$\mathrm{d}(T', \mathrm{Id}_{\mathcal{H}}) = \|\ln(T')\| = \ln(\|T'\|) = \ln(s(S)).$$
This establishes the proposition. ☐

Let us now explain how we pass from arbitrary fixed points to unitarisers.

Lemma 3.38. Let π be a unitarisable representation of a group G. If $T \in \mathcal{P}(\mathcal{H})^G$, then there exists $S \in U(\pi)$ so that
$$\ln(s(S)) \leq \mathrm{d}(T, \mathrm{Id}_{\mathcal{H}}).$$

Proof. Let $T \in \mathcal{P}(\mathcal{H})^G$. Applying Lemma 3.23, we find that $S = T^{1/2}$ is in $U(\pi)$. But now we can apply Lemma 3.37 to find a fixed point T' with
$$\mathrm{d}(T', \mathrm{Id}_{\mathcal{H}}) = \ln(s(S))$$
and additionally the proof of Lemma 3.37 gives us the exact procedure to follow to recover T'. In particular, T' has spectral symmetry. Henceforth, we are left to show that
$$\mathrm{d}(T', \mathrm{Id}_{\mathcal{H}}) \leq \mathrm{d}(T, \mathrm{Id}_{\mathcal{H}}). \tag{10}$$
The left-hand side is $\ln(\|T'\|)$, while the right-hand side is $\max(\ln(\|T\|), \ln(\|T^{-1}\|))$. To establish (10), we then distinguish two cases:
$$\text{(i) } \|T^{-1}\| \leq \|T\| \text{ and (ii) } \|T^{-1}\| \geq \|T\|.$$

(i) In that case, we must see $\ln(\|T'\|) \leq \ln(\|T\|)$, or equivalently $\|T'\| \leq \|T\|$ (as $\ln\colon (0,\infty) \longrightarrow \mathbb{R}$ is increasing). We reformulate this inequality using the definition of S and the construction of T' coming from the proof of 3.37:
$$\begin{aligned}\|T'\| \leq \|T\| &\iff \|T'\| \leq \|T^{1/2}(T^{1/2})^*\| \\ &\iff \lambda^2\|S\|^2 \leq \|S\|^2 \\ &\iff \lambda^2 \leq 1 \\ &\iff \|S^{-1}\| \leq \|S\|\end{aligned}$$

But this last inequality holds, since

$$\|S^{-1}\| = \|(T^{1/2})^{-1}\| = \|T^{-1}\|^{1/2} \leq \|T\|^{1/2} = \|T^{1/2}\| = \|S\|.$$

using (ii) and (iii) of Corollary 1.44. This proves (10) in the case (i).

(ii) Now assume that $\|T^{-1}\| \geq \|T\|$. This time, we must show that $\ln(\|T'\|) \leq \ln(\|T^{-1}\|)$, or equivalently $\|T'\| \leq \|T^{-1}\|$. We proceed exactly as in the previous case, and we get

$$\begin{aligned} \|T'\| \leq \|T^{-1}\| &\iff \|T'\| \leq \|T^{-1/2}(T^{-1/2})^*\| \\ &\iff \lambda^2\|S\|^2 \leq \|T^{-1/2}\|^2 \\ &\iff \lambda^2\|S\|^2 \leq \|S^{-1}\|^2 \\ &\iff \frac{\|S^{-1}\|}{\|S\|}\|S\|^2 \leq \|S^{-1}\|^2 \\ &\iff \|S\| \leq \|S^{-1}\|. \end{aligned}$$

Here again, this last inequality holds since

$$\|S\| = \|T^{1/2}\| = \|T\|^{1/2} \leq \|T^{-1}\|^{1/2} = \|T^{-1/2}\| = \|S^{-1}\|.$$

This proves (10) in the case (ii), and finishes the proof. □

We can now combine these two lemmas to obtain the following.

Proposition 3.39. Let π be a unitarisable representation of a group G, and let $S \in U(\pi)$ be a smallest unitariser. Then there exists a fixed point $T' \in \mathcal{P}(\mathcal{H})^G$ so that

$$\mathrm{d}(T', \mathrm{Id}_{\mathcal{H}}) = \mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathrm{Id}_{\mathcal{H}}) = \ln(s(S)).$$

Proof. Let then π be a unitarisable representation of a group G. From Lemma 3.38, it follows that

$$\inf_{T\in\mathcal{P}(\mathcal{H})^G} \mathrm{d}(T, \mathrm{Id}_{\mathcal{H}}) \geq \inf_{S\in U(\pi)} \ln(s(S))$$

and from Lemma 3.37 we get

$$\inf_{T\in\mathcal{P}(\mathcal{H})^G} \mathrm{d}(T, \mathrm{Id}_{\mathcal{H}}) \leq \inf_{S\in U(\pi)} \ln(s(S)).$$

Thus the two infimum coincide. Now, choose a smallest unitariser S of π. Apply Lemma 3.37 to get a fixed point T' so that

$$\mathrm{d}(T', \mathrm{Id}_{\mathcal{H}}) = \ln(s(S)).$$

As S minimizes sizes of unitarisers of π, it also minimizes the logarithm of the sizes of unitarisers of π, so

$$\mathrm{d}(T', \mathrm{Id}_{\mathcal{H}}) = \ln(s(S)) = \inf_{\tilde{S}\in U(\pi)} \ln(\tilde{S}) = \inf_{T\in\mathcal{P}(\mathcal{H})^G} \mathrm{d}(T, \mathrm{Id}_{\mathcal{H}}) = \mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathrm{Id}_{\mathcal{H}})$$

and the claim follows. □

This result translates Pisier's theorem geometrically.

Corollary 3.40. Let G be a unitarisable group.
There exist universal constants $C \in \mathbb{R}$, $\alpha > 0$, depending only on G, so that for any uniformly bounded representation π of G, one has

$$\mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathrm{Id}_{\mathcal{H}}) = \mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathcal{O}_{\mathrm{Id}_{\mathcal{H}}}) \leq C + \frac{\alpha}{2}\mathrm{diam}(\pi).$$

Proof. Fix G a unitarisable group, and consider the two constants $\alpha, K > 0$ given by Theorem 3.36. Let π be a uniformly bounded representation of G. The first equality is a direct consequence of the invariance of the metric d under the action of $\mathrm{Aut}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$ (Proposition 2.6):

$$\begin{aligned}\mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathcal{O}_{\mathrm{Id}_{\mathcal{H}}}) &= \inf_{g \in G,\, T \in \mathcal{P}(\mathcal{H})^G} \mathrm{d}(T, \theta_\pi(g, \mathrm{Id}_{\mathcal{H}})) \\ &= \inf_{g \in G,\, T \in \mathcal{P}(\mathcal{H})^G} \mathrm{d}(\theta_\pi(g^{-1}, T), \mathrm{Id}_{\mathcal{H}}) \\ &= \inf_{T \in \mathcal{P}(\mathcal{H})^G} \mathrm{d}(T, \mathrm{Id}_{\mathcal{H}}) \\ &= \mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathrm{Id}_{\mathcal{H}}).\end{aligned}$$

Let now S be a smallest unitariser for π. By Proposition 3.39, $\mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathrm{Id}_{\mathcal{H}})$ equals $\ln(s(S))$. Hence

$$\mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathrm{Id}_{\mathcal{H}}) = \ln(s(S)) \leq \ln(K|\pi|^\alpha) = \ln(K) + \alpha \ln(|\pi|) = \ln(K) + \frac{\alpha}{2}\mathrm{diam}(\pi)$$

using $s(S) \leq K|\pi|^\alpha$ for the upper bound and Proposition 3.31 for the last equality. We then set $C := \ln(K)$ and we are done. □

In [25], Pisier showed that a discrete group G is amenable if and only if Theorem 3.36 holds with $K = 1$ and $\alpha = 2$. Putting these values in Corollary 3.40, we obtain that a group G is amenable if and only if

$$\mathrm{d}(\mathcal{P}(\mathcal{H})^G, \mathcal{O}_{\mathrm{Id}_{\mathcal{H}}}) \leq \mathrm{diam}(\pi)$$

for any uniformly bounded representation π of G. Hence, letting

$$\begin{aligned}X_\pi &:= \{T \in \mathcal{P}(\mathcal{H}) : \mathrm{d}(T, \mathcal{O}_{\mathrm{Id}_{\mathcal{H}}}) \leq \mathrm{diam}(\mathcal{O}_{\mathrm{Id}_{\mathcal{H}}})\} \\ &= \{T \in \mathcal{P}(\mathcal{H}) : \mathrm{d}(T, \mathcal{O}_{\mathrm{Id}_{\mathcal{H}}}) \leq \mathrm{diam}(\pi)\}\end{aligned}$$

we obtain another characterization of amenability.

Corollary 3.41. A group G is amenable if and only if $X_\pi \cap \mathcal{P}(\mathcal{H})^G \neq \emptyset$ for every uniformly bounded representation π of G.

With this characterization, Dixmier's problem is turned as:

Is it true that $X_\pi \cap \mathcal{P}(\mathcal{H})^G$ is not empty, for every uniformly bounded representation of a unitarisable group G?

A. General topology

In this part we proceed to recalling general background material from general topology.

The first definition we may recall is that of a *topology* on a set.

Definition A.1. Let X be a set.
A topology on X is a collection τ of subsets of X so that

(i) $\emptyset, X \in \tau$.

(ii) If $\{U_i\}_{i\in I}$ is a family of elements of τ, then $\bigcup_{i\in I} U_i \in \tau$.

(iii) If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.

A set equipped with a topology τ is called a *topological space*, and the elements of τ are called the *open* subsets of X.

Example A.2. (i) If X is a set, then $\tau_{\text{disc}} = \mathcal{P}_{\text{s}}(X)$ is a topology on X, called the *discrete* topology. For this topology, all subsets of X are open. Likewise, the family $\tau_{\text{triv}} = \{\emptyset, X\}$ also forms a topology, called the *trivial* topology.

(ii) If τ_1 and τ_2 are both topologies on X, then $\tau_1 \cap \tau_2$ is a topology on X.

(iii) If (X, τ_X) is a topological space, and $Y \subset X$, the collection

$$\tau_Y := \{U \cap Y : U \in \tau_X\}$$

is a topology on Y. Indeed, as $\emptyset = \emptyset \cap Y$, $Y = X \cap Y$ and $\emptyset, X \in \tau_X$, we see that $\emptyset, Y \in \tau_Y$. If $V_1, V_2 \in \tau_Y$, we may write $V_1 = U_1 \cap Y$, $V_2 = U_2 \cap Y$ for some $U_1, U_2 \in \tau_X$ and it follows that

$$V_1 \cap V_2 = (U_1 \cap Y) \cap (U_2 \cap Y) = (U_1 \cap U_2) \cap Y.$$

As τ_X is a topology, $U_1 \cap U_2 \in \tau_X$ and we deduce $V_1 \cap V_2 \in \tau_Y$. Lastly, if $\{V_i\}_{i\in I}$ is a collection of elements of τ_Y, we write $V_i = U_i \cap Y$ with $U_i \in \tau_X$ for each $i \in I$ and thus

$$\bigcup_{i\in I} V_i = \bigcup_{i\in I}(U_i \cap Y) = \left(\bigcup_{i\in I} U_i\right) \cap Y.$$

Since τ_X is a topology, we see that $\bigcup_{i\in I} U_i \in \tau_X$, whence $\bigcup_{i\in I} V_i \in \tau_Y$, as wanted. Hence τ_Y is a topology on Y, called the *induced* topology.

If τ_1 and τ_2 are two topologies on a set X, we say that τ_1 is *smaller* than τ_2, or that τ_2 is *larger* than τ_1, if $\tau_1 \subset \tau_2$. This relation is an order relation on the set of topologies of X.

Let (X, τ) be a topological space. A subset $F \subset X$ is *closed* if $X \setminus F$ is open, and for $x \in X$ a subset $V \subset X$ is called a *neighbourhood* of x if there exists an open set $U \subset X$ so that $x \in U \subset V$.

From the definition, an open set is a neighbourhood of any of its elements. Conversely, if $U \subset X$ is a neighbourhood of any of its elements, then for any $x \in U$, we find an open set $U_x \subset X$ so that $x \in U_x \subset U$. This implies that U itself is open, as

$$U = \bigcup_{x \in U} U_x.$$

For $A \subset X$, its *interior* is the subset A° defined as

$$A^\circ := \{x \in X : A \text{ is a neighbourhood of } x\}$$

and its *closure* is

$$\overline{A} := \{x \in X : X \setminus A \text{ is not a neighbourhood of } x\}.$$

Directly from the definitions, we have $A^\circ \subset A \subset \overline{A}$, and we may define the *boundary* of A as $\partial A := \overline{A} \setminus A^\circ$. Lastly, a subset $A \subset X$ is called *dense* in X if $\overline{A} = X$.

Now we know the objects, we turn to the morphisms of the theory.

Definition A.3. Let (X, τ_X) and (Y, τ_Y) be two topological spaces.
A map $f: X \longrightarrow Y$ is continuous if $f^{-1}(U) \in \tau_X$ for all $U \in \tau_Y$.

Here are examples that follow from the definition.

Example A.4. (i) For any topological space (X, τ_X), the map $\mathrm{Id}_X: (X, \tau_X) \longrightarrow (X, \tau_X)$ is continuous. If τ_1 and τ_2 are different topologies on X, $\mathrm{Id}_X: (X, \tau_1) \longrightarrow (X, \tau_2)$ is continuous if and only if $\tau_2 \subset \tau_1$.

(ii) Any constant map is continuous. Indeed if $f: X \longrightarrow Y$ is so that $f(x) = y_0$ for any $x \in X$ and some $y_0 \in Y$, then for $U \subset Y$ open, $f^{-1}(U)$ is either empty or equal to X, and these subsets are open with respect to any topology on X.

(iii) If X carries its discrete topology, any map $f: X \longrightarrow Y$ is continuous. Likewise, any map between X and Y is continuous if Y carries its trivial topology.

(iv) The composition of two continuous maps is a continuous map. Suppose indeed that $f: (X, \tau_X) \longrightarrow (Y, \tau_Y)$, $g: (Y, \tau_Y) \longrightarrow (Z, \tau_Z)$ are both continuous and fix $U \in \tau_Z$. We have

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

and the continuity of g ensures that $g^{-1}(U) \in \tau_Y$. Continuity of f now ensures that $f^{-1}(g^{-1}(U)) \in \tau_X$, whence $g \circ f$ is continuous.

As the definition of continuity involves inverse images of open sets, and as inverse images are compatible with the complement, a map $f: X \longrightarrow Y$ between two topological spaces is continuous if and only if $f^{-1}(F)$ is closed in X for any closed subset $F \subset Y$.

Continuity can also be formulated locally, via neighbourhoods of points.

Proposition A.5. Let (X, τ_X), (Y, τ_Y) be two topological spaces.
A map $f : X \longrightarrow Y$ is continuous if and only if, for any $x \in X$, if V is a neighbourhood of $f(x) \in Y$, $f^{-1}(V)$ is a neighbourhood of $x \in X$.

Proof. Suppose first that f is continuous. Let $x \in X$, and fix V a neighbourhood of $f(x)$ in Y. Hence there is $U \in \tau_Y$ so that $f(x) \in U \subset V$. This implies $x \in f^{-1}(U) \subset f^{-1}(V)$, and as f is continuous, $f^{-1}(U) \in \tau_X$. This proves that $f^{-1}(V)$ is a neighbourhood of x, as wanted.

Conversely, fix $U \in \tau_Y$, and let $x \in f^{-1}(U)$. Then $f(x) \in U$, and U is a neighbourhood of $f(x)$, so by the assumption we deduce that $f^{-1}(U)$ is a neighbourhood of x. This proves that $f^{-1}(U)$ is a neighbourhood of any of its elements, which means it is an open set in X. Hence f is continuous, and we are done. □

We can now introduce the isomorphisms of the theory.

Definition A.6. Let X, Y be two topological spaces.
A map $f : X \longrightarrow Y$ is a homeomorphism if f is continuous, bijective, and f^{-1} is continuous.

When there is a homeomorphism between X and Y, we say that X and Y are *homeomorphic*, and we note $X \cong Y$. This is an equivalence relation on the class of topological spaces.

We now turn to study a huge class of topological spaces, that plays a central role in general topology, namely metric spaces.

Definition A.7. Let X be a set.
A metric on X is a map $\mathrm{d}_X : X \times X \longrightarrow [0, \infty)$ so that

(i) $\mathrm{d}_X(x, y) = 0$ if and only if $x = y$.

(ii) $\mathrm{d}_X(x, y) = \mathrm{d}_X(y, x)$ for any $x, y \in X$.

(iii) $\mathrm{d}_X(x, z) \leq \mathrm{d}_X(x, y) + \mathrm{d}_X(y, z)$ for any $x, y, z \in X$.

Property (ii) above is called the *symmetry* of d_X, while (iii) is the *triangle inequality*. A set X equipped with a metric is called a *metric space*.

Given a metric d_X on a set X, we denote $B_{\mathrm{d}_X}(x, r)$ the *ball* of radius $r > 0$ centered at $x \in X$, defined as

$$B_{\mathrm{d}_X}(x, r) := \{y \in X : \mathrm{d}_X(x, y) < r\}.$$

Example A.8. (i) If X is a set, the map $\mathrm{d}\colon X\times X \longrightarrow [0,\infty)$ sending (x,y) to 0 if $x=y$ and 1 otherwise, is a metric, called the *discrete* metric.

(ii) If $X=\mathbb{R}^n$, $n\geq 1$, and $p\in[1,\infty)$, we can define a metric d_p on $\mathbb{R}^n$ by the formula

$$\mathrm{d}_p(x,y) := \left(\sum_{i=1}^n |x_i-y_i|^p\right)^{1/p}$$

for $x=(x_1,\ldots,x_n), y=(y_1,\ldots,y_n)\in\mathbb{R}^n$. The case $p=2$ corresponds to the euclidean metric. We can also define d_p for $p=\infty$, by

$$\mathrm{d}_\infty(x,y) := \max_{1\leq i\leq n}|x_i-y_i|$$

for all $x=(x_1,\ldots,x_n), y=(y_1,\ldots,y_n)\in\mathbb{R}^n$.

A metric d_X on X induces a topology τ_{d_X} on X, by declaring that a subset $U\subset X$ is in τ_{d_X} if for any $x\in U$, there exists $r(x)>0$ so that $B_{\mathrm{d}_X}(x,r(x))\subset U$.

For this topology, if $x\in X$ and $r>0$, the ball $B_{\mathrm{d}_X}(x,r)$ is open: if $y\in B_{\mathrm{d}_X}(x,r)$, then $r' := r-\mathrm{d}_X(x,y)>0$, and $B_{\mathrm{d}_X}(y,r')\subset B_{\mathrm{d}_X}(x,r)$. Indeed if $z\in B_{\mathrm{d}_X}(y,r')$ then $\mathrm{d}_X(z,y)<r'$ whence

$$\mathrm{d}_X(z,x)\leq \mathrm{d}_X(z,y)+\mathrm{d}_X(y,x) < r'+\mathrm{d}_X(x,y) = r$$

using the triangle inequality and the symmetry of d_X. $B_{\mathrm{d}_X}(x,r)$ is then called the *open ball* of radius $r>0$ around $x\in X$. Likewise, if $x\in X$ and $r>0$ the set

$$B'_{\mathrm{d}_X}(x,r) := \{y\in X : \mathrm{d}(x,y)\leq r\}$$

is closed in X. It is the *closed ball* of radius $r>0$ around $x\in X$.

Moreover, two metrics $\mathrm{d}_X, \mathrm{d}'_X$ on a set X are *equivalent* if there exists $c,c'>0$ so that

$$c\mathrm{d}_X(x,y)\leq \mathrm{d}'_X(x,y)\leq c'\mathrm{d}_X(x,y)$$

for any $x,y\in X$.

Proposition A.9. If d_X and d'_X are equivalent, then $\tau_{\mathrm{d}_X}=\tau_{\mathrm{d}'_X}$.

Proof. If the two metrics are equivalent and $c,c'>0$ are as above, we have

$$B_{\mathrm{d}'_X}(x,cr)\subset B_{\mathrm{d}_X}(x,r)\subset B_{\mathrm{d}'_X}(x,c'r)$$

for any $x\in X$ and $r>0$. Now fix $U\in\tau_{\mathrm{d}_X}$. For any $x\in U$, we find $r(x)>0$ so that $B_{\mathrm{d}_X}(x,r(x))\subset U$. By the left most inclusion above we deduce $B_{\mathrm{d}'_X}(x,cr(x))\subset U$, which implies $U\in\tau_{\mathrm{d}'_X}$. Hence $\tau_{\mathrm{d}_X}\subset\tau_{\mathrm{d}'_X}$. The same game using the other inclusion above shows $\tau_{\mathrm{d}'_X}\subset\tau_{\mathrm{d}_X}$, and thus $\tau_{\mathrm{d}_X}=\tau_{\mathrm{d}'_X}$. □

Now, let us prove the following characterization of continuity for maps between metric spaces.

Theorem A.10. Let $(X, \mathrm{d}_X), (Y, \mathrm{d}_Y)$ be two metric spaces.
A map $f : X \longrightarrow Y$ is continuous if and only if for any $x \in X$, for any $\varepsilon > 0$, there is $\delta > 0$ so that if $x' \in X$ satisfies $\mathrm{d}_X(x, x') < \delta$, then $\mathrm{d}_Y(f(x), f(x')) < \varepsilon$.

Proof. Suppose first that f is continuous, and fix $x \in X$, $\varepsilon > 0$. By Proposition A.5, this means that for any neighbourhood V of $f(x)$ in Y, $f^{-1}(V)$ is a neighbourhood of x in X. In particular, for $V = B_{\mathrm{d}_Y}(f(x), \varepsilon)$, $f^{-1}(V)$ is a neighbourhood of x in X. It follows that there exists $\delta > 0$ so that $B_{\mathrm{d}_X}(x, \delta) \subset f^{-1}(B_{\mathrm{d}_Y}(f(x), \varepsilon))$. This last inclusion exactly means that if $x' \in X$ is such that $\mathrm{d}_X(x, x') < \delta$, then $\mathrm{d}_Y(f(x), f(x')) < \varepsilon$, which proves one direction.

Conversely, fix $x \in X$ and $V \subset Y$ a neighbourhood of $f(x)$. This implies there exists $\varepsilon > 0$ with $B_{\mathrm{d}_Y}(f(x), \varepsilon) \subset V$. Using the assumption, we find $\delta > 0$ so that $\mathrm{d}_Y(f(x), f(x')) < \varepsilon$ if $\mathrm{d}_X(x, x') < \delta$, or equivalently so that

$$B_{\mathrm{d}_X}(x, \delta) \subset f^{-1}(B_{\mathrm{d}_Y}(f(x), \varepsilon)).$$

As $f^{-1}(B_{\mathrm{d}_Y}(f(x), \varepsilon)) \subset f^{-1}(V)$, we proved there exists $\delta > 0$ so that $x \in B_{\mathrm{d}_X}(x, \delta) \subset f^{-1}(V)$. We deduce that $f^{-1}(V)$ is a neighbourhood of x in X. As $x \in X$ was arbitrary, Proposition A.5 implies that f is continuous. □

In this formulation of continuity, the δ we found after fixing $\varepsilon > 0$ and $x \in X$ may depend on both $\varepsilon > 0$ and $x \in X$. It is sometimes useful to find a *uniform* δ, the same for any point x in X, and that depends only on the particular $\varepsilon > 0$ we fixed beforehand. This leads to the following terminology: a map $f : (X, \mathrm{d}_X) \longrightarrow (Y, \mathrm{d}_Y)$ between two metric spaces is *uniformly continuous* if for any $\varepsilon > 0$, there is $\delta > 0$ such that if $x, x' \in X$ satisfy $\mathrm{d}_X(x, x') < \delta$, then $\mathrm{d}_Y(f(x), f(x')) < \varepsilon$.

Two remarks are in order here. Firstly, from this definition, any uniformly continuous map is continuous. Secondly, and in contrast with the notion of continuity, uniform continuity is a metric notion, and has no natural analog for more general topological spaces.

If $k > 0$, an application $f : (X, \mathrm{d}_X) \longrightarrow (Y, \mathrm{d}_Y)$ between two metric spaces is called *k−Lipschitz* if

$$\mathrm{d}_Y(f(x), f(x')) \leq k\mathrm{d}_X(x, x')$$

for any $x, x' \in X$. Such a map is automatically uniformly continuous, as for a fixed $\varepsilon > 0$, it suffices to choose $\delta := \frac{\varepsilon}{k} > 0$ in the above definition. In particular, k−Lipschitz maps are continuous.

If $(X, \mathrm{d}_X), (Y, \mathrm{d}_Y)$ are two metric spaces, an *isometry* between X and Y is a surjective map $f : X \longrightarrow Y$ so that

$$\mathrm{d}_Y(f(x), f(x')) = \mathrm{d}_X(x, x')$$

for all $x, x' \in X$. This condition implies that f is injective, because if $x, x' \in X$ are so that $f(x) = f(x')$, we get

$$\mathrm{d}_X(x, x') = \mathrm{d}_Y(f(x), f(x')) = 0$$

so $x = x'$, and uniformly continuous, because $1-$Lipschitz.

Here is a useful characterization of uniformly continuous maps.

Proposition A.11. Let $(X, \mathrm{d}_X), (Y, \mathrm{d}_Y)$ be two metric spaces.
A map $f : X \longrightarrow Y$ is uniformly continuous if and only if there exists an increasing function $F\colon [0,\infty) \longrightarrow [0,\infty]$ so that $F(0) = 0$, $F(t) \to 0$ as $t \to 0$, and

$$\mathrm{d}_Y(f(x), f(x')) \leq F(\mathrm{d}_X(x,x'))$$

for all $x, x' \in X$.

Proof. To begin, assume that such a function F exists, and let us show that f is uniformly continuous. Fix $\varepsilon > 0$. The fact that $F(t) \to 0$ when $t \to 0$ readily means there exists $\delta > 0$ (depending only on ε) so that $t < \delta$ implies $F(t) < \varepsilon$. In particular, for $x, x' \in X$ with $\mathrm{d}_X(x,x') < \delta$, we get

$$\mathrm{d}_Y(f(x), f(x')) \leq F(\mathrm{d}_X(x,x')) < \varepsilon$$

whence f is uniformly continuous.

Conversely, assume that f is uniformly continuous, and set

$$F(t) := \sup\{\mathrm{d}_Y(f(x), f(x')) : \mathrm{d}_X(x,x') \leq t\}$$

for all $t \in [0,\infty)$. From this definition, F is positive, increasing, satisfies $F(0) = 0$ and

$$\mathrm{d}_Y(f(x), f(x')) \leq F(\mathrm{d}_X(x,x'))$$

for all $x, x' \in X$. Lastly, let $\varepsilon > 0$. The uniform continuity of f provides $\delta > 0$ so that

$$x, x' \in X,\ \mathrm{d}_X(x,x') < \delta \Longrightarrow \mathrm{d}_Y(f(x), f(x')) < \varepsilon.$$

Now, if $t \in [0,\infty)$ is so that $t < \delta$, then

$$F(t) = \sup\{\mathrm{d}_Y(f(x), f(x')) : \mathrm{d}_X(x,x') \leq t\} < \varepsilon.$$

Thus $F(t) \to 0$ when $t \to 0$, concluding the proof. □

When f is uniformly continuous, a function F as in the proposition is often called a *modulus of continuity* of f.

Definition A.12. Let X be a metric space and $A \subset X$.
Its diameter is defined as

$$\mathrm{diam}(A) := \sup_{x,y \in A} \mathrm{d}_X(x,y)$$

and we say that A is bounded if $\mathrm{diam}(A) < \infty$.

Moreover, if $A, B \subset X$ and $x \in X$, the distance between x and A is defined as

$$\mathrm{d}_X(x, A) := \inf_{a \in A} \mathrm{d}_X(x, a)$$

and the distance between A and B is $\mathrm{d}_X(A, B) := \inf_{a \in A,\, b \in B} \mathrm{d}_X(a, b)$.

As we will see below, metric spaces have a lot of useful topological properties. This motivates to ask whether all topological spaces are in fact metric spaces, and if not, which spaces still have a topology arising from a metric.

Definition A.13. A topological space (X, τ_X) is called metrisable if there exists a metric d_X on X so that $\tau_X = \tau_{\mathrm{d}_X}$.

Example A.14. (i) Obviously, any metric space is metrisable.

(ii) A discrete space is metrisable, since the discrete metric induces the discrete topology.

The goal of the next definition is to provide an efficient way of putting a topology on a space without describing all open subsets.

Definition A.15. Let (X, τ_X) be a topological space.

(i) A subset $\mathcal{B} \subset \tau_X$ is a basis for τ_X if any element of τ_X is a union of elements of $\mathcal{B}$.

(ii) A subset $\mathcal{S} \subset \tau_X$ is a subbasis for τ_X if the set

$$\mathcal{B} := \{S_1 \cap \cdots \cap S_n : n \geq 1,\ S_1, \ldots, S_n \in \mathcal{S}\}$$

is a basis for τ_X.

Example A.16. (i) For any set X, $\mathcal{B} := \{\{x\} : x \in X\}$ is a basis for the discrete topology on X.

(ii) For any set X, $\mathcal{B} := \{X\}$ is a basis for the trivial topology on X.

(iii) If (X, d_X) is a metric space, the set

$$\mathcal{B} := \{B_{\mathrm{d}_X}(x, r) : x \in X, r > 0\}$$

is a basis for τ_{d_X}. Indeed, if $U \in \tau_{\mathrm{d}_X}$, then for any $x \in U$, there is $r(x) > 0$ so that $B_{\mathrm{d}_X}(x, r(x)) \subset U$. This implies

$$U = \bigcup_{x \in U} B_{\mathrm{d}_X}(x, r(x))$$

whence $\mathcal{B}$ is a basis.

A priori, if X is a set, an arbitrary subset of $\mathcal{P}_s(X)$ has no reason to be a basis or a subbasis for a topology on X. There are necessary and sufficient conditions to check this is the case. Here they are.

Proposition A.17. Let X be a set, and $\mathcal{B} \subset \mathcal{P}_s(X)$.
$\mathcal{B}$ is a basis for a topology on X if and only if the following hold:

(i) $X = \bigcup_{B \in \mathcal{B}} B$.

(ii) For any $B_1, B_2 \in \mathcal{B}$, for any $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ so that $x \in B \subset B_1 \cap B_2$.

In particular, $\mathcal{S} \subset \mathcal{P}_s(X)$ is a subbasis for a topology on X if and only if

$$X = \bigcup_{S \in \mathcal{S}} S.$$

Proof. Let us assume first that $\mathcal{B} \subset \mathcal{P}_s(X)$ is a basis for a topology on X, *i.e.* the collection τ of all unions of elements of $\mathcal{B}$ is a topology on X. In particular, $X \in \tau$, so X is a union of elements of $\mathcal{B}$, which establishes (i). To prove (ii), let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. In particular, $B_1, B_2 \in \tau$, which is closed under finite intersections, so $B_1 \cap B_2 \in \tau$. Thus $B_1 \cap B_2$ is a union of elements of B, and in particular we may choose $B \in \mathcal{B}$ so that $x \in B \subset B_1 \cap B_2$. This shows (ii).

Conversely, assume that (i) and (ii) holds. We show that the collection τ consisting of union of elements of $\mathcal{B}$ is a topology on X. Firstly, (i) immediately implies that $X \in \tau$, and by convention $\emptyset$ is the empty union, so $\emptyset \in \tau$. Point (ii) in Definition A.1 is obvious, as a union of union of elements of $\mathcal{B}$ is a union of elements of $\mathcal{B}$. We are thus left to show that τ is closed under finite intersections. Let then $U_1, U_2 \in \tau$. By definition of τ, we may write

$$U_1 = \bigcup_{i \in I} B_i, \; U_2 = \bigcup_{j \in J} B'_j$$

for two arbitrary collections $\{B_i\}_{i \in I}$, $\{B'_j\}_{j \in J}$ of elements of $\mathcal{B}$. It follows that

$$U_1 \cap U_2 = \bigcup_{i \in I,\, j \in J} (B_i \cap B'_j).$$

Now fix $x \in U_1 \cap U_2$. Then there is $i \in I, j \in J$ with $x \in B_i \cap B'_j$, and assumption (ii) ensures there is $B(x) \in \mathcal{B}$ so that $x \in B(x) \subset B_i \cap B'_j$. We conclude that

$$U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} B(x)$$

which exhibits $U_1 \cap U_2$ as a union of elements of $\mathcal{B}$, *i.e.* $U_1 \cap U_2 \in \tau$. Hence τ is a topology on X, as claimed.

The last statement about subbasis is an immediate consequence of Definition A.15 and the equivalence criteria we just proved for basis. □

When $\mathcal{B} \subset \mathcal{P}_s(X)$ is a basis for a topology on X, we denote this topology as $\tau_{\mathcal{B}}$, and we say it is *generated* by $\mathcal{B}$.

A topological space is said to be *second countable* if its topology has a countable basis.

Basis and subbasis are useful to reduce the workload of many checks with topological spaces. For instance, if $\mathcal{S}$ is a subbasis for a topology τ_Y on a set Y, a map $f\colon (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is continuous if and only if $f^{-1}(S) \in \tau_X$ for any $S \in \mathcal{S}$, as unions and intersections of sets are preserved by inverse images. Likewise, if $\mathcal{B}$ and $\mathcal{B}'$ are two basis on a set X, $\tau_{\mathcal{B}} \subset \tau_{\mathcal{B}'}$ if and only if for any $x \in X$ and any $B \in \mathcal{B}$ containing x, there is $B' \in \mathcal{B}'$ so that $x \in B' \subset B$.

As for continuity, we can also consider local basis, around a fixed point $x \in X$ in a topological space.

Definition A.18. Let X be a topological space and $x \in X$.
A set $\mathcal{B}_x$ of neighbourhoods of x is a basis of neighbourhoods for x if, for any neighbourhood V of x, there is $B \in \mathcal{B}_x$ so that $x \in B \subset V$.

Let us illustrate this definition in our running examples.

Example A.19. (i) If X is a discrete space, then $\mathcal{B}_x := \{\{x\}\}$ is a basis of neighbourhoods for $x \in X$.

(ii) If X is a trivial space, $\mathcal{B}_x := \{X\}$ is a basis of neighbourhoods for any $x \in X$.

(iii) If (X, d_X) is a metric space and $x \in X$, then

$$\mathcal{B}_x := \{B_{\mathrm{d}_X}(x, r) : r > 0\}$$

is a basis of neighbourhoods for $x \in X$.

Furthermore, we say that X is *first countable* if any $x \in X$ has a countable basis of neighbourhoods. As a corollary to Example A.19(iii), we get that any metric space (X, d_X) is first countable, because

$$\mathcal{B}_x := \left\{B_{\mathrm{d}_X}\big(x, \frac{1}{n}\big) : n \geq 1\right\}$$

is a countable basis of neighbourhoods of x, for any $x \in X$.

Let us now turn to the concept of convergence in a topological space, extending the one we know for sequences of real numbers. Recall that a *sequence* $(x_n)_{n \in \mathbb{N}}$ in a set X is a map $\mathbb{N} \longrightarrow X$.

Definition A.20. Let (X, τ_X) be a topological space, and $x \in X$.
A sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in X if for all $U \in \tau_X$ with $x \in U$, there exists $N \in \mathbb{N}$ so that $n \geq N \Longrightarrow x_n \in U$.

If a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in X$, we denote $x_n \longrightarrow x$.

From this definition and the one of a basis of neighbourhoods for a point $x \in X$, it follows that if $\mathcal{B}_x$ is a basis of neighbourhoods for x, a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to x if and only if for any $B \in \mathcal{B}_x$, there exists $N \in \mathbb{N}$ so that $n \geq N \Longrightarrow x_n \in B$.

Likewise, if the topology τ_X comes from a subbasis $\mathcal{S}$, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in X$ if and only if for all $S \in \mathcal{S}$ with $x \in S$ there exists $N \in \mathbb{N}$ so that $x_n \in S$ for all $n \geq N$.

Example A.21. (i) Suppose that X is a discrete space. Letting $x \in X$ and applying Definition A.20 with the open set $U = \{x\}$, we see that if a sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$, there exists $N \in \mathbb{N}$ so that $x_n = x$ for all $n \in \mathbb{N}$, forcing the sequence to be in fact constant when n is large enough. Conversely, such sequences are convergent. Hence a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges if and only if it is eventually constant.

(ii) In sharp contrast, if X is equipped with its trivial topology, the definition of convergence is satisfied for any sequence $(x_n)_{n\in\mathbb{N}}$ and any point $x \in X$. Thus all sequences converge to all points in X.

(iii) Suppose that (X, d_X) is a metric space, and let $x \in X$. We saw that $\mathcal{B}_x := \{B_{\mathrm{d}_X}(x, \varepsilon) : \varepsilon > 0\}$ is a basis of neighbourhoods of x, so applying the above remark, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ so that $n \geq N \Longrightarrow \mathrm{d}_X(x_n, x) < \varepsilon$.

Point (ii) in this example illustrates a phenomenon we typically want to avoid, as in $\mathbb{R}$. We introduce then a new class of spaces.

Definition A.22. A topological space (X, τ_X) is Hausdorff if for any pair of distinct points $x, y \in X$, there exists $U, V \in \tau_X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

When such open sets exist for two points $x, y \in X$, we say they *separate* x and y.

As from the beginning, we first investigate this property in our favorite examples.

Example A.23. (i) Any discrete space X is Hausdorff: for $x \neq y \in X$, it suffices to consider $U = \{x\}$ and $V = \{y\}$.

(ii) On the other hand, if X carries its trivial topology, then X is Hausdorff if and only if $|X| \leq 1$.

(iv) If (X, d_X) is a metric space, it is Hausdorff. For $x \neq y \in X$, consider $U := B_{\mathrm{d}_X}(x, \varepsilon)$ and $V := B_{\mathrm{d}_X}(y, \varepsilon)$, where $\varepsilon := \frac{\mathrm{d}_X(x,y)}{2} > 0$. U and V are both open, contain x and y respectively, and are disjoint. This shows in fact that any metrisable space is Hausdorff.

In Hausdorff spaces, there is indeed uniqueness of limits of convergent sequences.

Proposition A.24. Let X be a Hausdorff space.
If $(x_n)_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ and to $y \in X$, then $x = y$.

Proof. We prove the contrapositive. Assume that $x \neq y$ and that $x_n \longrightarrow x$. We show it cannot converge to y. By hypothesis, X is Hausdorff, and $x \neq y$, so we find two open sets U and V so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since $x_n \longrightarrow x$, there exists $N \in \mathbb{N}$ so that $n \geq N \Longrightarrow x_n \in U$. The fact that U and V are disjoint then implies that $x_n \notin V$ for all $n \geq N$, and in particular $(x_n)_{n\in\mathbb{N}}$ cannot converge to $y \in X$. This finishes the proof. □

We will see below that the converse holds under the additional assumption of first countability. Before proving this result, we introduce one last concept related to convergent sequences and continuity.

Definition A.25. Let X, Y be two topological spaces.
A map $f : X \longrightarrow Y$ is sequentially continuous in $x \in X$ if for any sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \rightarrow x$ in X, $f(x_n) \rightarrow f(x)$ in Y.

Moreover, we call f *sequentially continuous* if it is sequentially continuous in $x \in X$ for any $x \in X$.

The following lemma is a direct consequence of the definitions.

Lemma A.26. Let X, Y be two topological spaces.
If $f : X \longrightarrow Y$ is continuous, then f is sequentially continuous.

Proof. Suppose that f is continuous, and let $x \in X$. Consider a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \rightarrow x$, and an open set $V \subset Y$ so that $f(x) \in V$. It follows from the continuity of f that $f^{-1}(V)$ is open in X. Now $x \in f^{-1}(V)$, so by the definition of convergence, there exists $N \in \mathbb{N}$ so that $x_n \in f^{-1}(V)$ for any $n \geq N$, and thus $f(x_n) \in V$ for any $n \geq N$. This establishes that $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in Y, whence f is sequentially continuous in x. As this holds for any $x \in X$, f is sequentially continuous. □

Here also, it is natural to ask whether the converse holds or not. The answer will also come from a countability assumption on X. To prove this result, we require an auxiliary lemma.

Lemma A.27. Let X be a first countable topological space.
Any $x \in X$ admits a countable and decreasing basis of open neighbourhoods, *i.e.* a countable basis of neighbourhoods $\mathcal{B}_x = \{B_n\}_{n\in\mathbb{N}}$ so that B_n is open in X and $B_{n+1} \subset B_n$ for every $n \in \mathbb{N}$.
Moreover, any sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \in B_n$ for all $n \in \mathbb{N}$ converges to x.

Proof. Let $x \in X$. As X is first countable, choose a countable basis of neighbourhoods $\mathcal{B}_x = \{B''_n\}_{n\in\mathbb{N}}$ of x. For any $n \in \mathbb{N}$, there is a open set $B'_n \subset B''_n$ containing x, and thus $B_n := B'_1 \cap \cdots \cap B'_n$, $n \in \mathbb{N}$, provides a countable basis of open neighbourhoods of x which is decreasing.

For the second claim, fix a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in B_n$ for any $n \in \mathbb{N}$. As we just proved, we can assume that the sequence $\{B_n\}_{n\in\mathbb{N}}$ is decreasing, and thus for any $m \in \mathbb{N}$, we see that $x_n \in B_n \subset B_m$ for all $n \geq m$. By the remark following Definition A.20 we conclude that $x_n \to x$ as $n \to \infty$, and we are done. □

Without any delay, let us provide answers to the two previous questions.

Theorem A.28. Let X be a first countable topological space, and Y be a topological space. Then

(i) X is Hausdorff if and only if any convergent sequence in X has a unique limit.

(ii) A map $f : X \longrightarrow Y$ is continuous if and only if it is sequentially continuous.

Proof. (i) One direction is Proposition A.24. For the converse, assume that X is not Hausdorff. This means we may find two points $x \neq y$ in X so that for any pair of open sets U, V with $x \in U$ and $y \in V$, $U \cap V \neq \emptyset$. As X is first countable, Lemma A.27 ensures we can pick two countable decreasing bases of open neighbourhoods $\mathcal{B}_x = \{B_n(x)\}_{n\in\mathbb{N}}$, $\mathcal{B}_y = \{B_n(y)\}_{n\in\mathbb{N}}$ for x and y respectively. Then $B_n(x) \cap B_n(y) \neq \emptyset$ for all $n \in \mathbb{N}$, so choose $x_n \in B_n(x) \cap B_n(y)$, $n \in \mathbb{N}$. By Lemma A.27, $x_n \to x$ and $x_n \to y$, and this concludes the proof of (i).

(ii) One direction is Lemma A.26, and for the converse we proceed as in (i). Assume that f is not continuous. By definition, we find an open set $U \subset Y$ such that $f^{-1}(U)$ is not open in X. This implies we may find $x \in f^{-1}(U)$ with the property that for any neighbourhood V of x, $V \not\subset f^{-1}(U)$. In particular, for $\mathcal{B}_x = \{B_n\}_{n\in\mathbb{N}}$ a countable decreasing basis of open neighbourhoods of x, we must have $B_n \not\subset f^{-1}(U)$, and we can then choose $x_n \in B_n \setminus f^{-1}(U)$, for any $n \in \mathbb{N}$. By Lemma A.27, $x_n \to x$, but we do not have $f(x_n) \to f(x)$. This shows that f is not sequentially continuous, and finishes the proof of the theorem. □

Let us now focus on the notion of *compactness* for topological spaces. To formulate the definition, we need a terminology: if X is a topological space, an *open covering* of X is a family $\{U_i\}_{i\in I}$ of open sets in X so that

$$X = \bigcup_{i\in I} U_i.$$

We say an open covering $\{U_i\}_{i\in I}$ admits a *finite subcovering* if there exists a finite subset $J \subset I$ so that

$$X = \bigcup_{j\in J} U_j.$$

Definition A.29. A topological space X is compact if any open covering of X admits a finite subcovering.

If $Y \subset X$ is endowed with the topology induced by that of X, it is compact if for any family $\{U_i\}_{i\in I}$ of open sets in X so that $Y \subset \bigcup_{i\in I} U_i$, there exists $J \subset I$ finite so that $Y \subset \bigcup_{j\in J} U_j$.

Example A.30. (i) If X has only a finite number of open sets, then X is compact. In particular, if X is finite, any topology on X has at most $|\tau_{\mathrm{disc}}| = |\mathcal{P}_{\mathrm{s}}(X)| = 2^{|X|}$ open sets, so X is compact for any of its topologies.

(ii) It follows from (i) that any set X with its trivial topology $\{\emptyset, X\}$ is compact.

(iii) On the other hand, a discrete space X is compact if and only if it is finite. Indeed, if it is finite, it is compact by (i). If it is infinite, $\{\{x\} : x \in X\}$ is an open covering of X without any finite subcovering.

(iv) The space $X = \mathbb{R}$ with its usual topology is not compact, since $\{(n-1, n+1) : n \in \mathbb{Z}\}$ is an open covering of $\mathbb{R}$ without any finite subcovering.

We provide now a list of general properties of compactness.

Proposition A.31. Let X be a compact topological space, and let $Y \subset X$. If Y is closed in X, then Y is compact.

Proof. Consider a family $\{U_i\}_{i\in I}$ of open sets of X with

$$Y \subset \bigcup_{i\in I} U_i.$$

As Y is closed in X, $X \setminus Y$ is open in X, so $\{X \setminus Y\} \cup \{U_i\}_{i\in I}$ is an open covering of X. Since X is compact, we find $i_1, \dots, i_m \in I$ so that

$$X \subset (X \setminus Y) \cup U_{i_1} \cup \cdots \cup U_{i_m}.$$

Thus we conclude that $Y \subset U_{i_1} \cup \cdots \cup U_{i_m}$, and Y is compact. □

The next proposition shows that the converse holds if the ambient space is Hausdorff.

Proposition A.32. Let X be a topological Hausdorff space, and let $Y \subset X$. If Y is compact, then Y is closed in X.

Proof. Let $x \in X \setminus Y$. For each $y \in Y$, $x \neq y$, and since X is Hausdorff, we find two open sets U_y, V_y of X so that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. The collection $\{V_y\}_{y \in Y}$ is an open covering of Y, which is compact, and we therefore find $y_1, \ldots, y_m \in Y$ so that

$$Y \subset V_{y_1} \cup \cdots \cup V_{y_m}.$$

Consider now $U := U_{y_1} \cap \cdots \cap U_{y_m}$. It is an open set in X as an intersection of finitely many open sets in X. It contains x as $x \in U_{y_i}$, for all $i = 1, \ldots, m$. Lastly, one has

$$U \cap Y = U \cap (V_{y_1} \cup \cdots \cup V_{y_m}) = \bigcup_{i=1}^{m} (U \cap V_{y_i}) \subset \bigcup_{i=1}^{m} (U_{y_i} \cap V_{y_i}) = \emptyset$$

using that $U_{y_i} \cap V_{y_i} = \emptyset$ for all $i = 1, \ldots, m$. Hence $U \subset X \setminus Y$. Thus $X \setminus Y$ is a neighbourhood of any of its elements. This implies it is an open set in X, and Y is therefore closed in X. □

The result to come ensures that compact spaces enjoy the *finite intersection property*.

Proposition A.33. Let X be a topological space. The following are equivalent.

(i) X is compact.

(ii) If $\{F_i\}_{i \in I}$ is a collection of closed subsets of X so that $\bigcap_{i \in I} F_i = \emptyset$, then there exists a finite subset $J \subset I$ so that $\bigcap_{j \in J} F_j = \emptyset$.

(iii) If $\{F_i\}_{i \in I}$ is a collection of closed subsets of X so that $\bigcap_{j \in J} F_j \neq \emptyset$ for any finite subset $J \subset I$, then $\bigcap_{i \in I} F_i \neq \emptyset$.

Proof. (ii) $\iff$ (iii) is clear since one statement is the contrapositive of the other. We are thus left to prove (i) $\iff$ (ii).

(i) $\implies$ (ii) : Suppose X compact, and take $\{F_i\}_{i \in I}$ a collection of closed subsets of X so that $\bigcap_{i \in I} F_i = \emptyset$. For all $i \in I$, let $U_i = X \setminus F_i$, which is open in X since F_i is closed. One has then

$$X = \emptyset^c = \left(\bigcap_{i \in I} F_i \right)^c = \bigcup_{i \in I} U_i$$

and since X is compact, there exists $J \subset I$ finite such that $X = \bigcup_{j \in J} U_j$. But then

$$\emptyset = X^c = \left(\bigcup_{j \in J} U_j \right)^c = \bigcap_{j \in J} U_j^c = \bigcap_{j \in J} F_j$$

which establishes (ii).

(ii) $\Longrightarrow$ (i) : Take $\{U_i\}_{i\in I}$ an open covering of X, so $X = \bigcup_{i\in I} U_i$. This implies $\emptyset = \bigcap_{i\in I} U_i^c$, and U_i^c is closed for all $i \in I$. By hypothesis, there exists $J \subset I$ finite so that $\emptyset = \bigcap_{j\in J} U_j^c$, and we get

$$X = \bigcup_{j\in J} U_j$$

which means X is compact, as desired. □

From the following theorem one can prove that compactness is a *topological invariant*, *i.e.* is preserved by homeomorphisms.

Theorem A.34. Let X be a compact topological space, and Y a topological space. If $f : X \longrightarrow Y$ is continuous, then $f(X) \subset Y$ is compact.

Proof. Let $\{U_i\}_{i\in I}$ be an open covering of $f(X) \subset Y$, *i.e.* a family of open subsets of Y so that

$$f(X) \subset \bigcup_{i\in I} U_i.$$

As f is continuous, $f^{-1}(U_i)$ is open in X for all $i \in I$. Moreover we have the equalities

$$X = f^{-1}(f(X)) \subset f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i)$$

so $\{f^{-1}(U_i)\}_{i\in I}$ is an open covering of X. By compactness of X, there exists $i_1, \dots, i_m \in I$ so that

$$X = f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_m})$$

and this implies $f(X) \subset U_{i_1} \cup \cdots \cup U_{i_m}$. Hence $f(X)$ is compact, as announced. □

We close this part by defining and studying *sequential compactness*, which is closely related to compactness. For this, recall that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in a topological space X, a *subsequence* of $(x_n)_{n\in\mathbb{N}}$ is a sequence of the form $(x_{\varphi(n)})_{n\in\mathbb{N}}$ where $\varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ is a strictly increasing map.

Definition A.35. A topological space X is sequentially compact if any sequence in X has a convergent subsequence.

Example A.36. (i) A finite space X is sequentially compact. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $|X| < \infty$, it must contain a constant subsequence, which is convergent for any topology on X.

(ii) A discrete space X is sequentially compact if and only if it is finite. Indeed if X is finite, it is sequentially compact by (i). Conversely, if X is infinite, there is a sequence in X consisting of distinct elements, and such a sequence cannot have a convergent subsequence, because convergent sequences in a discrete space are eventually constant (by Example A.21(i)).

(iii) The space $X = \mathbb{R}$ with its standard topology is not sequentially compact, as for instance the sequence $x_n = n$, $n \in \mathbb{N}$, does not have a convergent subsequence.

Here is our main result on sequential compactness.

Theorem A.37. If X is compact and first countable, then X is sequentially compact.

Proof. Let us assume that X is first countable and compact. Fix a sequence $(x_n)_{n\in\mathbb{N}}$ in X. We first claim that $(x_n)_{n\in\mathbb{N}}$ has an *accumulation point*, *i.e.* there exists $y \in X$ so that for any open set $U \subset X$ with $y \in U$, U contains infinitely many terms of the sequence $(x_n)_{n\in\mathbb{N}}$.

We prove this claim by contradiction. Suppose that for any $y \in X$, there is an open set $U_y \subset X$ and an integer N_y so that $y \in U_y$ and $x_n \notin U_y$ for all $n \geq N_y$. The family $\{U_y\}_{y\in X}$ forms an open covering of X, which is compact, and we thus find $y_1, \dots, y_m \in X$ with $X = U_{y_1} \cup \cdots \cup U_{y_m}$. By assumption, for each $i = 1, \dots, n$, U_{y_i} can contain only finitely many terms of the sequence $(x_n)_{n\in\mathbb{N}}$, which has thus only finitely many terms, a contradiction. Hence $(x_n)_{n\in\mathbb{N}}$ has an accumulation point $y \in X$.

As X is first countable, we can choose a countable basis of neighbourhoods $\mathcal{B}_y = \{B_n\}_{n\in\mathbb{N}}$ of y, and we may assume B_n to be open for all $n \in \mathbb{N}$ and the sequence to be decreasing, by Lemma A.27. As y is an accumulation point for $(x_n)_{n\in\mathbb{N}}$, B_n contains infinitely many terms of the sequence, for all $n \in \mathbb{N}$. We can then choose $(x_{\varphi(n)})_{n\in\mathbb{N}}$ a subsequence of $(x_n)_{n\in\mathbb{N}}$ with $x_{\varphi(n)} \in B_n$ for all $n \in \mathbb{N}$. By Lemma A.27, this sequence converges to $y \in X$, which concludes the proof of the theorem. □

As metric spaces are first countable, we deduce from this result that compact metric spaces are sequentially compact.

To end this appendix, we describe two useful methods to create topologies, on a set or on a product of sets.

The first method is the following: let X be a set, and let $\mathcal{F} := \{f : X \longrightarrow Y_f\}$ be a family of applications with source X. Assume that Y_f is a topological space for any $f \in \mathcal{F}$. We define the *initial topology* $\sigma(X, \mathcal{F})$ on X to be the topology generated by the subbasis

$$\mathcal{B}_{\mathcal{F}} := \{f^{-1}(U) : f \in \mathcal{F},\ U \subset Y_f \text{ open}\}.$$

From this definition, it appears that $\sigma(X, \mathcal{F})$ is the smallest topology on X so that each $f \in \mathcal{F}$ is continuous. This observation provides a characterization to describe continuous maps with target space X.

Proposition A.38. Let X be a set endowed with an initial topology $\sigma(X, \mathcal{F})$. Let Z be a topological space. Then a map $g : Z \longrightarrow X$ is continuous if and only if $f \circ g \colon Z \longrightarrow Y_f$ is continuous for any $f \in \mathcal{F}$.

Proof. If g is continuous, $f \circ g$ is continuous for any $f \in \mathcal{F}$ as a composition of continuous maps (by Example A.4(iv)). Conversely, suppose that $f \circ g$ is continous for all $f \in \mathcal{F}$. It is enough to prove that $g^{-1}(V)$ is open in Z for any $V \in \mathcal{B}_{\mathcal{F}}$. Fix then such a $V \in \mathcal{B}_{\mathcal{F}}$, that we write as $V = f^{-1}(U)$ for some $f \in \mathcal{F}$ and some open set U in Y_f. Then

$$g^{-1}(V) = g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U)$$

and since U is open in Y_f and $f \circ g$ is continuous by assumption, it follows that $g^{-1}(V)$ is open in Z, concluding the proof. □

Another advantage of initial topologies is the description of convergence of sequences.

Proposition A.39. Let X be a set endowed with an initial topology $\sigma(X, \mathcal{F})$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and $x \in X$. Then $x_n \to x$ as $n \to \infty$ if and only if $f(x_n) \to f(x)$ as $n \to \infty$, for any $f \in \mathcal{F}$.

Proof. Suppose first that $x_n \to x$. Each $f \in \mathcal{F}$ is continuous, in particular sequentially continuous by Lemma A.26, whence $f(x_n) \to f(x)$. Conversely, suppose $f(x_n) \to f(x)$. Fix $V \in \mathcal{B}_{\mathcal{F}}$ with $x \in V$, and write it $V = f^{-1}(U)$ for some $f \in \mathcal{F}$ and $U \subset Y_f$ open. As $x \in V$, we have that $f(x) \in U$, and since $f(x_n) \to f(x)$, there exists $N \in \mathbb{N}$ so that $f(x_n) \in U$ for all $n \geq N$. Hence $x_n \in f^{-1}(U) = V$ for all $n \geq N$, and by the remark following Definition A.20, this implies that $x_n \to x$, as announced. We are done. □

For the second method, fix an arbitrary collection $(X_i)_{i \in I}$ of non-empty sets. Recall that the product set $\prod_{i \in I} X_i$ is the set of all collections of the form $(x_i)_{i \in I}$ with $x_i \in X_i$ for every $i \in I$. More precisely

$$\prod_{i \in I} X_i := \left\{ x \colon I \longrightarrow \bigcup_{i \in I} X_i : x(i) \in X_i \text{ for any } i \in I \right\}.$$

Assume that X_i is endowed with a topology τ_{X_i}, for any $i \in I$. Consider then the subset $\mathcal{S} \subset \mathcal{P}_{\mathrm{s}}(\prod_{i \in I} X_i)$ defined as

$$\mathcal{S} := \left\{ \pi_j^{-1}(U_j) \subset \prod_{i \in I} X_i : j \in I,\ U_j \in \tau_{X_j} \right\}$$

where $\pi_j \colon \prod_{i \in I} X_i \longrightarrow X_j$ is the natural projection on the j−th component of the product. By definition, elements of $\mathcal{S}$ are of the form

$$\pi_j^{-1}(U_j) = U_j \times \prod_{i \in I \setminus \{j\}} X_i$$

with $j \in I$ and $U_j \in \tau_{X_j}$. The collection $\mathcal{S}$ satisfies the condition of Proposition A.17, and thus is a subbasis for a topology on $\prod_{i\in I} X_i$. The corresponding topology is called the *product topology* on $\prod_{i\in I} X_i$, and the corresponding basis is

$$\mathcal{B} = \left\{ \prod_{j\in J} U_j \times \prod_{i\in I\setminus J} X_i : J \subset I \text{ finite}, \ U_j \in \tau_{X_j} \text{ for any } j \in J \right\}.$$

From this construction, we see that the product topology is the smallest topology on $\prod_{i\in I} X_i$ so that any projection π_j is continuous.

Proposition A.40. Let $(X_i)_{i\in I}$ be a collection of topological spaces. For any topological space X and any family of maps $\{f_i \colon X \longrightarrow X_i : i \in I\}$, there exists a unique map $f \colon X \longrightarrow \prod_{i\in I} X_i$ so that $\pi_i \circ f = f_i$ for every $i \in I$. Moreover, f is continuous if and only if f_i is continuous for every $i \in I$.

Proof. For the first statement, it suffices to set

$$\begin{aligned} f \colon X &\longrightarrow \prod_{i\in I} X_i \\ x &\longmapsto (f_i(x))_{i\in I}. \end{aligned}$$

For the second statement, if f is continuous, then $f_i = \pi_i \circ f$ is continuous for every $i \in I$, as the composition of continuous maps is a continous map. Conversely, suppose that f_i is continuous for every $i \in I$. Let V be an element of the subbasis for the product topology on $\prod_{i\in I} X_i$. By definition, V takes the form $V = \pi_j^{-1}(U_j)$ where $j \in I$ and U_j is open in X_j. Then

$$f^{-1}(V) = f^{-1}(\pi_j^{-1}(U_j)) = (\pi_j \circ f)^{-1}(U_j)$$

and as $\pi_j \circ f$ is continuous and U_j is open in X_j, we conclude that $f^{-1}(V)$ is open in X, whence f is continuous. This achieves the proof. □

Let I be a set. When $X_i = X$ for any $i \in I$, the product $\prod_{i\in I} X_i$ is denoted X^I and is merely the set of functions from I to X. We can easily describe the convergence in such a space for the product topology.

Proposition A.41. Let I be a set and let X be a topological space. A sequence $(f_n)_{n\in\mathbb{N}}$ converges to f in X^I if and only if $(f_n(i))_{i\in I}$ converges to $f(i)$ in X for any $i \in I$.

Proof. We saw earlier that convergence of sequences in a space can be formulated using only elements of the subbasis of the topology on the space. An element of the subbasis of the product topology on X^I has the form $\pi_i^{-1}(U)$ for some $i \in I$ and some open subset $U \subset X$, and as usual π_i is the natural projection on the $i-$th copy of X in the product. Then $(f_n)_{n\in\mathbb{N}}$ converges to f if and only if for every $i \in I$ and open subset $U \subset X$ so that $f \in \pi_i^{-1}(U)$, there is $N \in \mathbb{N}$ so that $f_n \in \pi_i^{-1}(U)$, *i.e.* if and only if for every $i \in I$ and open subset $U \subset X$ so that $f(i) \in U$, there is $N \in \mathbb{N}$ so that $f_n(i) \in U$. This last condition precisely means that the sequence $(f_n(i))_{n\in\mathbb{N}}$ converges to $f(i)$ in X, for every $i \in I$. This proves the proposition. □

Because of this description of the convergence in X^I, one often refers to the product topology on X^I as the topology of *pointwise convergence*.

An important feature of the product topology is that it preserves the Hausdorff property.

Proposition A.42. Let I be a set and $(X_i)_{i\in I}$ a collection of topological spaces. Then $\prod_{i\in I} X_i$ is Hausdorff if and only if X_i is Hausdorff, for all $i \in I$.

Proof. If the product is Hausdorff and $j \in I$, write $X_j = \pi_j(\prod_{i\in I} X_i)$ and use that the Hausdorff property is preserved by continuous maps to see that X_j is Hausdorff.

Conversely, assume that each X_i is Hausdorff. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be two distinct elements. Then, there is $j \in I$ so that $x_j \neq y_j \in X_j$, and as the latter is Hausdorff, we find $U_j, V_j \subset X_j$ two disjoint open sets with $x_j \in U_j, y_j \in V_j$. Now the sets

$$U_j \times \prod_{i\in I\setminus\{j\}} X_i,\ V_j \times \prod_{i\in I\setminus\{j\}} X_i$$

are open in $\prod_{i\in I} X_i$ by definition of the product topology, disjoint since $U_j \cap V_j = \emptyset$, and the first one contains $(x_i)_{i\in I}$, the second one contains $(y_i)_{i\in I}$. This proves that the product is Hausdorff and finishes the proof. □

Another particularly important property of the product topology is that it preserves compactness:

Tychonoff's theorem. Let $(X_i)_{i\in I}$ be a collection of topological spaces.
Then $\prod_{i\in I} X_i$ is compact if and only if X_i is compact for every $i \in I$.

Clearly, if the product is compact, each X_i must be compact, applying Theorem A.34 with $f = \pi_i$. The heart of the theorem really lies in the converse statement. We will not present a proof here, as it goes far beyond the aim of this appendix, and we refer to [3, theorem A.2.1].

B. Amenability of groups

The aim of this appendix is to provide the reader with basic definitions and results about amenability for groups. Those will be used in our study of unitarisability in Chapter 3.

We start by recalling several equivalent ways of defining amenability for a group, without proofs. Then we establish various stability properties for the class of amenable groups, that we illustrate through numerous examples. Along the way we also derive the non-amenability of non-abelian free groups.

The basic concept required to formulate amenability is that of means.

Definition B.1. Let X be a set.
A mean on X is a map $\mu\colon \mathcal{P}_{\mathrm{s}}(X) \longrightarrow [0,1]$ so that $\mu(X) = 1$ and

$$\mu(A \sqcup B) = \mu(A) + \mu(B)$$

for all disjoint subsets $A, B \subset X$.

With words, a mean on a set X is a *finitely* additive probability measure.

Example B.2. Suppose $X \neq \emptyset$ and fix $x \in X$. The *Dirac mass at x* is defined as

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for all $A \in \mathcal{P}_{\mathrm{s}}(X)$. Directly, $\delta_x \in \mathcal{M}(X)$.

We denote $\mathcal{M}(X)$ the set of means on X. It is a convex subset of the $\mathbb{R}$−vector space $\mathbb{R}^{\mathcal{P}(X)}$, as if $\mu, \eta \in \mathcal{M}(X)$ and $t \in [0,1]$, then $((1-t)\mu + t\eta)(X) = (1-t)\mu(X) + t\eta(X) = 1 - t + t = 1$ and

$$\begin{aligned} ((1-t)\mu + t\eta)(A \sqcup B) &= (1-t)\mu(A \sqcup B) + t\eta(A \sqcup B) \\ &= (1-t)\mu(A) + t\eta(A) + (1-t)\mu(B) + t\eta(B) \\ &= ((1-t)\mu + t\eta)(A) + ((1-t)\mu + t\eta)(B) \end{aligned}$$

for all disjoint subsets $A, B \subset X$. Thus $(1-t)\mu + t\eta \in \mathcal{M}(X)$. From there, an induction proves that any convex combination of means on X is a mean on X.

As for probability measures, there are several properties inherited from the definition. If μ is as in Definition B.1, it satisfies

(i) $\mu(\emptyset) = 0$.

(ii) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$, for all $A, B \subset X$.

(iii) $\mu\left(\bigsqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$ for all $A_1, \ldots, A_n \subset X$.

(iv) $\mu(A) \leq \mu(B)$ for all $A, B \subset X$ such that $A \subset B$.

Observe that the inclusion $\mathcal{M}(X) \subset [0,1]^{\mathcal{P}_s(X)}$ naturally endows $\mathcal{M}(X)$ with a topology, induced by the product topology on $[0,1]^{\mathcal{P}_s(X)}$. This is the topology of pointwise convergence, by Proposition A.41, *i.e.* a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X)$ converges to $\mu \in \mathcal{M}(X)$ if and only if

$$\mu_n(A) \longrightarrow \mu(A)$$

as $n \to \infty$, for all $A \subset X$.

The following fact will be essential for our purposes.

Lemma B.3. Let X be a set. Then $\mathcal{M}(X)$ is compact.

Proof. A subbasis for the topology of $\mathcal{M}(X)$ is given by

$$\left\{ O \times \prod_{\mathcal{P}_s(X)\setminus\{A\}} \mathbb{R} \ : \ O \subset \mathbb{R} \quad \text{open}, \ A \subset X \right\}.$$

The complement of a set of this form is $(\mathbb{R} \setminus O) \times \prod_{\mathcal{P}_s(X)\setminus\{A\}} \mathbb{R}$. Hence we can construct closed subsets of $\mathbb{R}^{\mathcal{P}_s(X)}$ by fixing finitely many $A_1, \ldots, A_n \in \mathcal{P}_s(X)$, choosing any closed subset $C \subset \mathbb{R}^{\{A_1,\ldots,A_n\}} \simeq \mathbb{R}^n$, and considering

$$C \times \prod_{\mathcal{P}_s(X)\setminus\{A_1,\ldots,A_n\}} \mathbb{R}.$$

Now we turn to the proof of the lemma. Indeed, note to begin that in fact

$$\mathcal{M}(X) \subset [0,1]^{\mathcal{P}_s(X)}$$

and the latter is compact by Tychonoff's theorem. By Proposition A.31 it is thus enough to prove that $\mathcal{M}(X)$ is closed in $[0,1]^{\mathcal{P}_s(X)}$. Expanding the definition, we have

$$\begin{aligned} \mathcal{M}(X) &= \{\mu \in [0,1]^{\mathcal{P}_s(X)} : \mu(X) = 1, \ \mu(A \sqcup B) = \mu(A) + \mu(B) \ \forall A, B \in \mathcal{P}_s(X)\} \\ &= \{\mu \in [0,1]^{\mathcal{P}_s(X)} \ : \ \mu(X) = 1\} \\ &\cap \bigcap_{A,B\in\mathcal{P}_s(X),\ A\cap B=\emptyset} \left\{ \mu \in [0,1]^{\mathcal{P}_s(X)} \ : \ \mu(A \sqcup B) = \mu(A) + \mu(B) \right\}. \end{aligned}$$

All sets in this writing of $\mathcal{M}(X)$ are closed by the observation above: the first one corresponds to the choice $A_1 = X$, $C = \{1\}$. The second one, for A and B fixed with $A \cap B = \emptyset$, corresponds to $A_1 = A$, $A_2 = B$, $A_3 = A \sqcup B$ and $C = \{(x, y, z) \in \mathbb{R}^3 : x + y = z\}$. Therefore $\mathcal{M}(X)$ is closed in $[0,1]^{\mathcal{P}_s(X)}$ as an intersection of closed sets. This concludes our proof. □

The first formal definition of amenability was proposed by Von Neumann in 1929, after its study of the Banach-Tarski paradox.

Definition B.4. A group G is amenable if there exists a mean on G so that $\mu(gA) = \mu(A)$ for all $A \subset G$ and $g \in G$.

Here the notation gA stands for the translate of A under the action of G on itself by left multiplication: $gA := \{ga : a \in A\}$.

When $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subset G$, we say that μ is G*−invariant*. With this terminology, a group G is amenable if and only if it carries a G−invariant mean.

Example B.5. Any finite group G is amenable, since the normalized counting measure, defined as $\mu(A) := \frac{|A|}{|G|}$ for any $A \subset G$, is a G−invariant mean.

Going beyond finite groups, it is in general *not* possible to provide an exact formula for an invariant mean on a group. We must then proceed differently, and a fruitful idea is to take advantage of the compactness of $\mathcal{M}(G)$ and consider accumulation points of well-chosen sequences. To that end, the next remark is crucial.

Remark B.6. A set map $f : X \longrightarrow Y$ induces a well-defined map $f_* : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$, setting

$$(f_*(\mu))(B) := \mu(f^{-1}(B)), \ \mu \in \mathcal{M}(X), \ B \subset Y.$$

If $\mathcal{O} \times \prod_{\mathcal{P}_s(Y)\setminus\{A\}} \mathbb{R}$ is an element of the subbasis for the product topology on $\mathcal{M}(Y)$, with $\mathcal{O} \subset \mathbb{R}$ open and $A \subset Y$, we have

$$f_*^{-1}\Bigg(\mathcal{O} \times \prod_{\mathcal{P}_s(Y)\setminus\{A\}} \mathbb{R}\Bigg) = \mathcal{O} \times \prod_{\mathcal{P}_s(X)\setminus\{f^{-1}(A)\}} \mathbb{R}$$

which is open for the product topology on $\mathcal{M}(X)$. Hence f_* is continuous.

We are now ready to give our first example of an infinite amenable group.

Theorem B.7. The group $\mathbb{Z}$ is amenable.

Proof. For $n \geq 1$, consider $\mu_n := \frac{1}{n}\sum_{j=1}^{n} \delta_j$. Any convex combination of means is a mean, so $\mu_n \in \mathcal{M}(\mathbb{Z})$ for every $n \geq 1$. By Lemma B.3, $\mathcal{M}(\mathbb{Z})$ is compact, so $(\mu_n)_{n\geq 1}$ has an accumulation point $\mu \in \mathcal{M}(\mathbb{Z})$ (by the proof of Theorem A.37). We now prove μ is a left invariant mean for the action $\mathbb{Z} \curvearrowright \mathbb{Z}$, *i.e.* we show that $g\mu = \mu$ for all $g \in \mathbb{Z}$. Writing $\mathbb{Z} = \langle u \rangle$ multiplicatively, it is enough to prove that $u\mu = \mu$. First, note that if $A \subset \mathbb{Z}$ and $j \in \mathbb{Z}$, then

$$u\delta_j(A) = \delta_j(u^{-1}A) = \begin{cases} 1 & \text{if } j \in u^{-1}A \\ 0 & \text{if } j \notin u^{-1}A \end{cases} = \begin{cases} 1 & \text{if } j+1 \in A \\ 0 & \text{if } j+1 \notin A \end{cases} = \delta_{j+1}(A)$$

so that $u\delta_j = \delta_{j+1}$. It follows that $u\mu_n = \dfrac{1}{n}\sum_{j=2}^{n+1}\delta_j$, and thus $u\mu_n - \mu_n = \frac{1}{n}(\delta_{n+1} - \delta_1)$ for all $n \geq 1$. This implies that

$$u\mu_n - \mu_n \longrightarrow 0 \tag{11}$$

as $n \to \infty$, in $\mathbb{R}^{\mathcal{P}_s(\mathbb{Z})}$. If, towards a contradiction, $u\mu \neq \mu$, we can separate these two points by disjoint neighbourhoods U and V, because $\mathbb{R}^{\mathcal{P}_s(\mathbb{Z})}$ is Hausdorff by Proposition A.42. Since μ is an accumulation point of $(\mu_n)_{n\geq 1}$ and $u\mu$ is an accumulation point of $(u\mu_n)_{n\geq 1}$, we can find infinitely many terms of the sequence $(\mu_n)_{n\geq 1}$ in U and infinitely many terms of $(u\mu_n)_{n\geq 1}$ in V. Since they are disjoint, this contradicts (11). Thus $u\mu = \mu$, and this finishes the proof. ◻

Above, the fact that $u\mu$ is an accumulation point of $(u\mu_n)_{n\geq 1}$ follows from the continuity of

$$\begin{aligned} \varphi_u \colon \mathcal{M}(\mathbb{Z}) &\longrightarrow \mathcal{M}(\mathbb{Z}) \\ \mu &\longmapsto u\mu \end{aligned}$$

and the continuity of φ_u follows from Remark B.6.

From this example, we will derive below numerous examples of amenable groups.

Before this, let us in contrast provide also the simpliest example of a non-amenable group: the non-abelian free group on two generators.

Recall that if S is a set, there exists a group called the *free group on S* and denoted F_S, satisfying the following universal property: for any group G and any map $f\colon S \longrightarrow G$, there is a unique group homomorphism $\tilde{f}\colon F_S \longrightarrow G$ so that $\tilde{f} \circ \iota = f$, where $\iota\colon S \hookrightarrow F_S$ is the natural inclusion of S in F_S ([4, theorem 1.5], [28, theorem 11.1]).

For any set S, F_S depends only on $|S|$ [28, theorem 11.4], up to isomorphism, and $|S|$ is therefore called the *rank* of F_S. We write F_d if $|S| = d$, and F_∞ if $|S| = |\mathbb{N}|$. The group F_0 is trivial, and F_1 is infinite cyclic. For $n \geq 2$, F_n is not abelian. Free groups play a central role in group theory, and more details on their properties can be found in [4, chapter 1], [28, chapter 11]. One particularly important result about them is the so called *Nielsen-Schreier theorem* ([4, theorem 1.15], [28, theorem 11.44]), stating that any subgroup of a free group is itself a free group.

For us, non-abelian free groups provide the other part of the spectrum, opposite to finite groups and $\mathbb{Z}$, as they are not amenable.

Theorem B.8. The group F_2 is not amenable.

Proof. Suppose for a contradiction that there exists an invariant mean μ on F_2. Write $F_2 = \{e\} \sqcup A_+ \sqcup A_- \sqcup B_+ \sqcup B_-$, where A_+ (resp. A_-) consists of reduced words starting with an a (resp. a^{-1}) and B_+ (resp. B_-) consists of reduced words starting with a b (resp. b^{-1}). Since the second letter of an element of A_+ can be an a, a b or a b^{-1}, multiplying this element by a^{-1} produces an element either of A_+, B_+ or B_-. Then

$$a^{-1}A_+ = \{e\} \sqcup A_+ \sqcup B_+ \sqcup B_-.$$

Properties of μ then imply

$$\mu(A_+) = \mu(a^{-1}A_+) = \mu(\{e\} \sqcup A_+ \sqcup B_+ \sqcup B_-) = \mu(\{e\}) + \mu(A_+) + \mu(B_+) + \mu(B_-)$$

and erasing $\mu(A_+)$ of both sides leaves us with $\mu(\{e\}) + \mu(B_+) + \mu(B_-) = 0$. Since μ takes positive values, this forces $\mu(\{e\}) = \mu(B_+) = \mu(B_-) = 0$. Likewise, we get $\mu(A_+) = \mu(A_-) = 0$. We conclude that

$$\begin{aligned} 1 = \mu(F_2) &= \mu(\{e\} \sqcup A_+ \sqcup A_- \sqcup B_+ \sqcup B_-) \\ &= \mu(\{e\}) + \mu(A_+) + \mu(A_-) + \mu(B_+) + \mu(B_-) \\ &= 0 \end{aligned}$$

which is absurd. Therefore such a μ cannot exist. □

As a matter of fact, the proof makes apparent an observation already done by Von Neumann: the existence of an invariant mean on a group is an obvious obstruction for this group to have a *paradoxical decomposition*. More precisely, if a group G is amenable, then there cannot exist $A_1, \dots, A_n, B_1, \dots, B_m \subset G$ non-empty and $g_1, \dots, g_n, h_1, \dots, h_m \in G$ so that

$$G = A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m = g_1A_1 \sqcup \cdots \sqcup g_nA_n = h_1B_1 \sqcup \cdots \sqcup h_mB_m.$$

What is much less obvious is that it is the only obstruction, namely if a group does not carry an invariant mean, it must have a paradoxical decomposition. This result is an outstanding theorem from Alfred Tarski, the proof of which can be found for instance in [4, section 14.3].

A group G acts on itself by left multiplication, and this action induces an action of G on $\mathcal{M}(G)$, by defining

$$(g \cdot \mu)(A) := \mu(g^{-1}A)$$

for every $\mu \in \mathcal{M}(G)$, $g \in G$ and $A \subset G$. Denoting $\mathcal{M}(G)^G$ the set of fixed points for this action, we see that a $G-$invariant mean is merely an element of $\mathcal{M}(G)^G$, *i.e.* G is amenable if and only if $\mathcal{M}(G)^G \neq \emptyset$.

Moreover, as proved earlier, $\mathcal{M}(G)$ is compact, and we therefore see that a sufficient condition for G to be amenable is to fix a point in every non-empty convex compact $G-$space in a locally convex topological vector space. It turns out that the converse holds as well, *i.e.* an amenable group fixes a point in every convex compact $G-$space [2, theorem G.1.7].

Lastly, here is a third characterization of amenability. For a group G, let

$$\mathcal{M}'(G) := \{m \in \ell^\infty(G)^* : m \geq 0,\ m(\mathbf{1}_G) = 1,\ \|m\| \leq 1\}$$

where $m \geq 0$ significates $m(f) \geq 0$ for every $f \geq 0$. Then G acts naturally on $\mathcal{M}'(G)$, by

$$(g \cdot m)(f) := m(g^{-1}f), \ m \in \mathcal{M}'(G), \ g \in G, \ f \in \ell^\infty(G).$$

Observe that an element $m \in \mathcal{M}'(G)$ gives rise to a mean $\mu_m \in \mathcal{M}(G)$ defined as

$$\mu_m(A) := m(\mathbf{1}_A)$$

for all $A \subset G$. One has indeed $\mu_m(G) = m(\mathbf{1}_G) = 1$ and μ_m is additive on disjoint subsets of G as

$$\mu_m(A \sqcup B) = m(\mathbf{1}_{A \sqcup B}) = m(\mathbf{1}_A + \mathbf{1}_B) = m(\mathbf{1}_A) + m(\mathbf{1}_B) = \mu_m(A) + \mu_m(B).$$

for $A, B \subset G$ disjoint. This correspondence can in fact be inverted, *i.e.* there is a bijection between $\mathcal{M}(G)$ and $\mathcal{M}'(G)$. Coupled with the equality

$$\mu_m(gA) = m(\mathbf{1}_{gA}) = m(g\mathbf{1}_A) = (g^{-1} \cdot m)(\mathbf{1}_A)$$

it follows that G has a $G-$invariant mean if and only if there exists $m \in \mathcal{M}'(G)$ so that $m(gf) = m(f)$ for every $g \in G$ and every $f \in \ell^\infty(G)$.

We shall summarize all these results in a theorem.

Theorem B.9. A group G is amenable if and only if one of the following equivalent conditions hold:

(i) G has a $G-$invariant mean.

(ii) G has no paradoxical decomposition.

(iii) G fixes a point in any non-empty convex compact $G-$space.

(iv) There exists $m \in \mathcal{M}'(G)$ so that $m(gf) = m(f)$ for every $g \in G$ and $f \in \ell^\infty(G)$.

Another characterization is due to Erling Følner, who formulated amenability in terms of *almost invariant sets*. More precisely, he showed in [17] that a group G is amenable if and only if for any $\varepsilon > 0$ and any finite subset $S \subset G$, there exists a finite subset $F \subset G$ so that

$$|sF \Delta F| < \varepsilon |F|$$

for all $s \in S$. Slight variations of this definition lead to introduce *Følner sequences* and *Følner nets* [4, section 14.4].

Later on, Hans Reiter also introduced its own version of amenability [27], in terms of $(S, \varepsilon)-$invariant vectors for the induced actions of a group G on the ℓ^p-spaces $\ell^p(G)$, $p \in [1, \infty)$, and defined the corresponding *Reiter properties* (R_p). We investigated these properties in [13], proving that (R_1) is equivalent to (R_2), and the same argument shows that (R_p) is equivalent to (R_q) for any $p, q \in [1, \infty)$. It is also showed in [13, proposition 2.6] that (R_2) is equivalent to a property related to the uniform convexity of $\ell^2(G)$ ([13, lemma 1.17], [4, proposition 14.33]).

Perhaps more surprisingly, amenability also relates to probabilistic phenomena on groups, with in particular the celebrated *Kesten's theorem*, established by Harry Kesten in [22] in 1959, that we also derived in [13, theorem 2.26].

For much more details on these groundbreaking results, their proofs, and the theme of amenability, we refer to [4, chapter 14], [2, appendix G] or [24].

In the rest of this appendix, we will see how to use the fixed-point characterization to establish the basic stability properties of the class of amenable groups. This way we get plenty of amenable groups, and the non-amenability of F_2 will also bring other examples of non-amenable groups.

Let us begin by recording the following.

Proposition B.10. Let G be an amenable group.

(i) If $H \leqslant G$, then H is amenable.

(ii) If $N \lhd G$, then G/N is amenable.

Proof. (i) See [4, theorem 14.9].

(ii) Let K be a convex compact $G/N-$space. This is naturally a $G-$space by letting $g \cdot x := (gN) \cdot x$ for all $g \in G$ and $x \in K$. As this $G-$space is convex compact, and G is amenable, we get a $G-$fixed point $y \in K$, which is by definition a $G/N-$fixed point for the initial action of G/N on K. Thus G/N is amenable, and (ii) is proved. □

Coupled with Theorem B.8, we deduce the following.

Corollary B.11. Any group containing a subgroup isomorphic to F_2 is not amenable. In particular, F_d is not amenable for all $d \geq 2$.

We also obtained in a different way the non-amenability of F_d, $d \geq 2$, in [13, corollary 2.32], via Kesten's theorem.

However, for our purposes, Proposition B.10 does not help us to extend the class of amenable groups because any subgroup or quotient of a finite group or $\mathbb{Z}$ is itself finite or isomorphic to $\mathbb{Z}$. To get bigger groups from old ones, one considers extensions of groups.

Definition B.12. Let G, Q be two groups, and $N \lhd G$.
We say that G is an extension of N by Q if $G/N = Q$.

This is the same as saying that there is a short exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

Example B.13. Consider the matrix group

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \ : \ a \in \mathbb{R}^*, b \in \mathbb{R} \right\}$$

also denoted Aff($\mathbb{R}$), and called the *affine group* on $\mathbb{R}$. Consider the two subgroups

$$N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \ : \ b \in \mathbb{R} \right\} \cong \mathbb{R}, \ Q := \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \ : \ a \in \mathbb{R}^* \right\} \cong \mathbb{R}^*$$

and the group homomorphism $\varphi \colon G \longrightarrow \mathbb{R}^*$ sending $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ to a. Then $N = \mathrm{Ker}(\varphi)$ is a normal subgroup of G, and as φ is surjective, it induces a group isomorphism $G/N \cong \mathbb{R}^*$. Thus G is an extension of N by Q.

Using the fixed point property, we can establish the following.

Theorem B.14. If G is an extension of N by Q, and N, Q are amenable, then G is amenable.

Proof. Let $K := \mathcal{M}(G)$. By our previous considerations, K is a convex compact $G-$space. In particular, it is a convex compact $N-$space and N being amenable, it follows that $K^N \neq \emptyset$. Writing

$$K^N = \bigcap_{n \in N} \{\mu \in K : n\mu = \mu\}$$

and observing that $\{\mu \in K : n\mu = \mu\}$ is closed in K for all $n \in N$, we get that K^N is closed in K. In particular K^N is compact since K is compact. Now the initial action of G on K restricts to K^N. Indeed if $g \in G$, $\mu \in K^N$ and $n \in N$, one computes that

$$n \cdot (g\mu) = g(g^{-1}ng \cdot \mu) = g\mu$$

since $g^{-1}ng \in N$, as N is normal in G, and $\mu \in K^N$. Since N acts trivially on K^N, this last action is in fact an action of Q, and its amenability implies that $(K^N)^Q \neq \emptyset$. This is equivalent to say that $K^G \neq \emptyset$, whence G is amenable. □

In particular, direct and semi-direct products of amenable groups are amenable. Henceforth, by Theorem B.7 and an induction on $d \geq 1$, $\mathbb{Z}^d$ is amenable for any $d \geq 1$.

For the next statements, recall that a group G is *finitely generated* if there exists a finite subset $S \subset G$ so that any element of G can be written as a product of elements of S or their inverses. In this case, we write $G = \langle S \rangle$.

Corollary B.15. Any finitely generated abelian group is amenable.

Proof. Let thus G be a finitely generated abelian group. Then there exists a finite group F and an integer $d \geq 1$ so that $G \cong \mathbb{Z}^d \times F$ (see *e.g.* [4, corollary 1.30]). Now $\mathbb{Z}^d$ is amenable, and F is amenable by Example B.5. Thus G is amenable as well. □

Theorem B.14 can also be used to prove the non-amenability of certain groups. For instance $\mathrm{SL}_2(\mathbb{Z})$ is an extension of its center $\{\pm I_2\}$ by the quotient $\mathrm{SL}_2(\mathbb{Z})/\{\pm I_2\} = \mathrm{PSL}_2(\mathbb{Z})$. As $\mathrm{SL}_2(\mathbb{Z})$ is not amenable, it follows that $\mathrm{PSL}_2(\mathbb{Z})$ is not amenable either.

The next operation on groups we consider is the *directed union*.

Definition B.16. Let G be a group, and $\mathcal{F}$ a collection of subgroups of G. We say $\mathcal{F}$ is directed if for any $H, H' \in \mathcal{F}$, there is $H'' \in \mathcal{F}$ such that $H, H' \leqslant H''$. Moreover, G is the directed union of $\mathcal{F}$ if $\mathcal{F}$ is directed and

$$G = \bigcup_{H \in \mathcal{F}} H.$$

As one might guess, amenability is also preserved when taking directed unions.

Theorem B.17. If G is the directed union of $\mathcal{F}$ and if any $H \in \mathcal{F}$ is amenable, then G is amenable.

Proof. Let $K := \mathcal{M}(G)$. We want to prove that $K^G \neq \emptyset$. As G is the directed union of $\mathcal{F}$, it is enough to show

$$\bigcap_{H \in \mathcal{F}} K^H \neq \emptyset.$$

As in the proof of Theorem B.14, K^H is closed for any $H \in \mathcal{F}$ and K is compact, so by Proposition A.33 it is in fact enough to show that

$$\bigcap_{i=1}^{n} K^{H_i} \neq \emptyset$$

for every $H_1, \ldots, H_n \in \mathcal{F}$. Fix then $H_1, \ldots, H_n \in \mathcal{F}$. As $\mathcal{F}$ is directed, we find $H \in \mathcal{F}$ so that $H_1, \ldots, H_n \leqslant H$, which implies

$$\bigcap_{i=1}^{n} K^{H_i} \supset K^H.$$

Since H is amenable, and K is convex compact, we deduce from Theorem B.9(iii) that $K^H \neq \emptyset$, finishing the proof. □

Corollary B.18. A group G is amenable if and only if all its finitely generated subgroups are amenable.

Proof. If G is amenable, all its subgroups are amenable by Proposition B.10. Conversely, suppose any finitely generated subgroup of G is amenable, and consider

$$\mathcal{F} := \{H \leqslant G : H \text{ is finitely generated}\}.$$

The collection $\mathcal{F}$ is directed, as if $H = \langle S \rangle$, $H' = \langle S' \rangle$ are both finitely generated, the subgroup $H'' = \langle S \cup S' \rangle$ is in $\mathcal{F}$ and contains both H and H' as subgroups. By assumption, any $H \in \mathcal{F}$ is amenable, and G is the directed union of $\mathcal{F}$, whence G is amenable by Theorem B.17. □

This result allows us to strenghthen Corollary B.15.

Corollary B.19. Any abelian group is amenable.

Proof. Suppose that G is an abelian group, and let $H \leqslant G$ be a finitely generated subgroup of G. Then H itself is a finitely generated abelian group, and thus is amenable by Corollary B.15. Henceforth, G has all its finitely generated subgroups amenable, and Corollary B.18 now implies that G is amenable. □

This way, we get for instance the amenability of $(\mathbb{Q}, +)$, even if the latter is not finitely generated. Likewise, we obtain the amenability of $(\mathbb{R}^n, +)$, for any $n \geq 1$.

Corollary B.20. Any solvable group is amenable.

Proof. A solvable group is obtained from the trivial group by doing finitely many extensions by abelian groups. As these are amenable, and as amenability is preserved by extensions, the claim follows. □

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